Computer Algebra

MSE

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Exam summary

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Summary created by Matthias Meyer. Note there might be errors!

Contents

Some parts are also available on the following webpage.

1 Newton Polynomische Interpolation

1.1 Collocation

Collocation = All measurement points are represented by a function, for example a polynomial. The polynomial

$$y(x) = p(x) = c_0 + c_1 x^1 + c_2 x^2 + \ldots + c_m x^m \quad \text{mit} \quad y(x_k) = p(x_k) = y_k \quad (k = 0, 1, \ldots, n)$$

results in a linear equation system with a degree of n+1.

1.2 Aitken-Neville Rekursionsformel

One way to solve this is by searching a polynomial formula as can be seen below. (It would also possible to generate a function by connecting the different data points with a straight line. ⇒ On a Computer this would take a lot of computational effort, since one would have a lot of if and else statements)

$$y(x) = p(x) = c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + \cdots + c_{m-1} x^{m-1} + c_m x^m \quad (c_0, c_1, \ldots \in \mathbb{R})$$

To get the solution of this polynomial, one has to solve the following equation system:

$$\begin{aligned}
y_0 &= c_0 + c_1 {x_0}^1 + c_2 {x_0}^2 + + c_3 x_0^3 + \cdots + c_{m-1} {x_0}^{m-1} + c_m {x_0}^m \\
y_1 &= c_0 + c_1 {x_1}^1 + c_2 {x_1}^2 + + c_3 {x_1}^3 + \cdots + c_{m-1} {x_1}^{m-1} + c_m {x_1}^m \\
&\vdots \\
y_n &= c_0 + c_1 {x_n}^1 + c_2 {x_n}^2 + + c_3 {x_n}^3 + \cdots + c_{m-1} {x_n}^{m-1} + c_m {x_n}^m
\end{aligned}$$

As one can see, this equation system gets really large and could be difficult to be solved on a microcontroller. But there exists a nice algorithm which makes solving this system easier, called Aitken-Neville recursion which divides the huge equation system in little parts.

$$p(x) = p_{0,1,2,\ldots,n-1,n}(x) = \frac{(x - x_0)\overbrace{p_{1,2,\ldots,n-1,n}}^{\text{green}}(x) - (x - x_n)\overbrace{p_{0,1,2,\ldots,n-1}}^{\text{red}}(x)}{(x_n - x_0)}$$

The formula above shows how the global interpolating polynomial is combined from the partial interpolation polynomials $p_{0,1,2,\ldots,n-1}(x)$and $p_{1,2,\ldots,n-1,n}(x)$. On this partial interpolation polynomials, one can again apply the formula until one ends up with only two datapoints.

1.3 Newton basis polynomials

Since the calculation with the Aitken-Neville recursion is quite tedious. Newton came up with another basis polynomial $\pi_k(x)$ with $k = 0, 1, 2, ..., n$ (Aitken-Neville recursion used $(1, x^1, x^2, ..., x^m)$)

$$\begin{aligned}
\pi_0(x) &= 1 \\
\pi_1(x) &= (x - x_0) \\
\pi_2(x) &= (x - x_0)(x - x_1) \\
&\vdots \\
\pi_k(x) &= (x - x_0)(x - x_1) \cdots (x - x_{k-1}) \\
&\vdots \\
\pi_n(x) &= (x - x_0)(x - x_1) \cdots (x - x_{n-1})
\end{aligned} \tag{1}$$

Which results in the final polynomial which can be seen in Equation 2

$$p(x) = a_0\pi_0(x) + a_1\pi_1(x) + a_2\pi_2(x) + \cdots + a_m\pi_m(x) \tag{2}$$

When one now writes the equation system one sees that this system is much easier to solve:

$$\begin{aligned}
y_0 &= a_0 \\
y_1 &= a_0 + a_1(x_1 - x_0) \\
y_2 &= a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\
&\vdots \\
y_n &= a_0 + a_1\pi_1(x_n) + a_2\pi_2(x_n) + \cdots + a_n\pi_n(x_n)
\end{aligned}$$

When one also applies the Aitken-Neville recursion it even get easier and independent of the order, since one just calculates divided differences. Below one can see the calculation of the first a_k terms.

$k = 0$	$y(x_0)$
$k = 1 : y(x_0, x_1)$	$\frac{y(x_1) - y(x_0)}{(x_1 - x_0)}$
$k = 2 : y(x_0, x_1, x_2)$	$\frac{y(x_1, x_2) - y(x_0, x_1)}{(x_2 - x_0)}$
$k = 3 : y(x_0, x_1, x_2, x_3)$	$\frac{y(x_1, x_2, x_3) - y(x_0, x_1, x_2)}{(x_3 - x_0)}$

Where $y(x_0, x_1, \ldots, x_k)$ is called the divided difference.

$$y(x_0, x_1, \ldots, x_k) = \frac{y(x_1, x_2, \ldots, x_k) - y(x_0, x_1, \ldots, x_{k-1})}{(x_k - x_0)} \quad (k = 0, 1, \ldots, n)$$

When the points have the same distance to each other, the formula gets even easier:

$$y(x_0, x_1, \ldots, x_k) = \frac{\Delta^k y_0}{h^k k!}$$

1.4 Example: Collocation polynomial

Given is the following dataset:

$$\{(0,1), (1,1), (2,2), (4,5)\} = \left\{\left(x_k, y_k\right) \mid k = 0, 1, \ldots, n = 3\right\}$$

Calculate the collocation polynomial.

One can do the calculation the following way:

$$\begin{array}{cccccc}
x_0 & y_0 & & & & \\
 & & \Delta y_0 & & & \\
x_1 & y_1 & & \Delta^2 y_0 & & \\
 & & \Delta y_1 & & \Delta^3 y_0 & \\
x_2 & y_2 & & \Delta^2 y_1 & & \Delta^4 y_0 \\
 & & \Delta y_2 & & \Delta^3 y_1 & \\
x_3 & y_3 & & \Delta^2 y_2 & & \\
 & & \Delta y_3 & & & \\
x_4 & y_4 & & & &
\end{array}$$

and therefore get the following result for the given data points:

x	y	π_1	π_2	π_3
$\underbrace{0}_{x_0}$	$\underbrace{1}_{a_0}$			
		$\frac{1-1}{1-0} = \underbrace{0}_{a_1}$		
$\underbrace{1}_{x_1}$	1		$\frac{1-0}{2-0} = \underbrace{\frac{1}{2}}_{a_2}$	
		$\frac{2-1}{2-1} = 1$		$\frac{\frac{1}{6}-\frac{1}{2}}{4-0} = -\frac{1}{12}$
2	2		$\frac{\frac{3}{2}-1}{4-1} = \frac{1}{6}$	
		$\frac{5-2}{4-2} = \frac{3}{2}$		
4	5			

According to Equation 2 one gets then the following result:

$$y(x) = p(x) = 1 + 0\pi_1(x) + \frac{1}{2}\pi_2(x) + \frac{-1}{12}\pi_3(x) =$$
$$1 + 0(x-0) + \frac{1}{2}(x-0)(x-1) + \frac{-1}{12}(x-0)(x-1)(x-2)$$

1.5 Further information

It does not depend on which data point one uses first and which one as last element. The resulting formula might look different (different a_k coefficients, but the last one is the same), but the result is exactly the same. Furthermore, the data points must not have the same spacing.

1.6 Problems

With a lot of data point the runge phenomenon occurs (oscillations with high frequencies and amplitudes towards the boundaries of the arguments range) To calculate the error which can be seen one can use the following formula:

$$y(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_{n-1})(x-x_n) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x)$$

Where ξ is a new data point.
$\frac{f^{(n+1)}(\xi)}{(n+1)!}$ can also be substituted by $\Rightarrow \mathrm{C} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$
It is a newton polynomial with one more argument. $\frac{f^{(n+1)}(\xi)}{(n+1)!}$ is a higher order derivative which we do not know at the moment. The formula can also be rewritten in Equation 3:

$$y(x) = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}}_{\mathrm{C}}\pi_{n+1}(x) + p(x) \tag{3}$$

When generalizing it one can also write Equation 4

$$y(x) - p(x) = \frac{\overbrace{y^{(d)}}^{\text{d'th derivative}}(\xi)}{d!}(x-x_0)^{d_0}(x-x_1)^{d_1}\cdots(x-x_n)^{d_n} \qquad (d = d_0 + d_1 + ...d_n) \tag{4}$$
$$x, \xi \in (\min x_i, \max x_i)_{i=0,1,...,n}$$

1.6.1 Error Calculation example

The model fucntion $y(x) = sin(\frac{1}{2}\pi x)$ has to be interpolated using the arguments $x = 0;1;2$ by a quadratic polynomial p(x), what is the error at the position $x = \frac{1}{2}$

x	y	π_1	π_2	π_3
0	$sin(\frac{1}{2}\pi 0) = 0$			
		$\frac{1-0}{1-0} = 1$		
1	$sin(\frac{1}{2}\pi 1) = 1$		$\frac{-1-1}{2-0} = -1$	
		$\frac{0-1}{2-1} = -1$		
2	$sin(\frac{1}{2}\pi 0) = 0$			

Error $= y(x) - p(x) = sin(\frac{1}{2}\pi x) - \underbrace{(-x^2 + 2x)}_{0+1(x-0)+1(x-0)(x-1)}$

2 Chebyshev arguments

To reduce the error mentioned above to a minimum one can use a chebyshev distribution of the arguments.

$$x_k = \cos\left(\frac{2k+1}{2(n+1)}\pi\right) \quad (k = 0, 1, \ldots, n)$$

2.0.1 Exercise 2 and 3 of excercise sheet 2 are important for the exam!

3 Hermite Interplation (Osculation)

When the problem of collocation, which can be found on the following post is extended by the requirement that certain given values of derivatives of order 0 up to some higher order k of the model function y must be met at some of the arguments $x_0, x_1, ..., x_n$ we end in an interpolation problem called osculation or **Hermite interpolation**. Furthermore note that one must use the modified newton polynomials as they can be seen in Equation 5, for the example provided there

3.1 Example: Hermite Interpolation and Error Callculation

We have railway track with the following points given: $(0;0), (2;1), (4;2)$ and also its derivatives. Now we have to search a polynomial p_2 which goes through point one and point two and fulfils it's derivatives. So we have the following conditions: $p_2(2) = 1, p_2'(2) = 1, p_2''(2) = 0$ and $p_2'(4) = 0, p_2(4) = 2, p_2''(4) = 0$

x	y	y'	$y''/2!$	$y'''/3!$	$y''''/4!$	$y'''''/5!$
$\underbrace{2}_{x_0}$	$\underbrace{1}_{a_0}$					
		$\frac{y^{(1)}(x_0)}{1!} = \underbrace{1}_{a_1 \text{ (given by ex.)}}$				
$\underbrace{2}_{x_0}$	1		$\frac{1}{2!} \cdot \underbrace{0}_{a_2 \text{ (given by ex.)}}$			
		$\frac{y^{(1)}(x_0)}{1!} = 1$		$\underbrace{\frac{-1}{8}}_{a_3}$		
$\underbrace{2}_{x_0}$	1		$-\frac{1}{4}$		$\underbrace{\frac{1}{16}}_{a_4}$	
		$\frac{2-1}{4-2} = \frac{1}{2}$		0		0
$\underbrace{4}_{x_1}$	2		$\frac{0-\frac{1}{2}}{4-2} = -\frac{1}{4}$		$\frac{1}{16}$	
		$\frac{y^{(1)}(x_1)}{1!} = 0$		$\frac{1}{8}$		
$\underbrace{4}_{x_1}$	2		$\frac{y^{(2)}(x_1)}{2!} = 0$			
		$\frac{y^{(1)}(x_1)}{1!} = 0$				
$\underbrace{4}_{x_1}$	2					

This lead to the following result: $p_2 = 1 \cdot \pi_0 + 1 \cdot \pi_1 + 0 \cdot \pi_2 - \frac{1}{8}\pi_3 + \frac{1}{16}\pi_4$ when using the modified newton polynomials as they can be seen in Equation 5.

$$\begin{aligned}
\pi_0 &= 1 \\
\pi_1 &= (x - x_0) = (x - 2) \\
\pi_2 &= (x - x_0)(x - x_0) = (x - 2)^2 \\
\pi_3 &= (x - x_0)(x - x_0)(x - x_0) = (x - 2)^3 \\
\pi_4 &= (x - x_0)(x - x_0)(x - x_0)(x - x_1) = (x - 2)^3 (x - 4) \\
\pi_5 &= (x - x_0)(x - x_0)(x - x_0)(x - x_1)(x - x_1) = (x - 2)^3 (x - 4)^2
\end{aligned} \tag{5}$$

The problem with this method it is that it is not guarateed to find a solution for this problem, when not all derivatives are given. One then has to increase the order of the polynomial.

Error With Equation 4 the error is then:

$$\frac{y^{(6)}(\xi)}{6!}(x-2)^3(x-4)^3$$

The maximum error therefore is: $\left(\max\left|y^{(6)}(\xi)\right|\right)\cdot\left(\max\left|(x-2)^3(x-4)^3\right|\right)\cdot\frac{1}{6!}$

3.2 Example 2

Compute two fourth-degree (!) polynomials, $p_1(x)$ and $p_2(x)$, meeting the constraints below:

$$p_1(0) = p_1'(0) = p_1''(0) = 0 \text{ and } p_1''(2) = 0$$

and

$$p_2(4) = 2, p_2'(4) = 0, p_2''(4) = 0 \text{ and } p_2''(2) = 0$$

Moreover, the two polynomials should meet smoothly at the point (2,1) without a crinkle (with a common tangential line) To solve this problem we introduce a new variable called a which defines the first derivative at point (2,1). When a is equal in both equations the meeting of the two polynomials is very smoothly. The first scheme looks like this:

x	y	y'	$y''/2!$	$y'''/3!$	$y''''/4!$	$y'''''/5!$
0	0					
		0				
0	0		$\frac{1}{2!}\cdot 0$			
		0		$\frac{1}{8}$		
0	0		$\frac{1}{4}$		$\frac{1}{16}$	
		$\frac{1-0}{2-0}=\frac{1}{2}$		$\lvert B\rvert$		$\lvert F\rvert$
2	1		$\lvert A\rvert$		$\lvert D\rvert$	
		a		$\lvert C\rvert$		
2	1		$\frac{1}{2!}\cdot 0$			
		a				
2	1					

Where

$$|A| = \frac{a-\frac{1}{2}}{2-0} = \frac{2a-1}{4}$$

$$|B| = \frac{|A|-\frac{1}{4}}{2-0} = \frac{2a-2}{8} = \frac{a-1}{4}$$

$$|D| = \frac{|B|-\frac{1}{8}}{2-a} = \frac{2a-3}{16}$$

$$|C| = \frac{0-|A|}{2-0} = \frac{1-2a}{8}$$

$$|E| = \frac{|C|-|B|}{2-c} = \frac{3-4a}{16}$$

$$|F| = \frac{|E|-|D|}{2-a} = \frac{3-3a}{16}$$

since p_1 must be of order 4 se conclude that $|\underline{\overline{F}}| = 0(!)$ and therefore $a = 1$

$$p_1 = 0 + 0 + 0 + \frac{1}{8}x^3 - \frac{1}{16}x^3(x-2) = \frac{1}{4}x^3 - \frac{1}{16}x^4$$

Now one can do the same thing for the next polynomial but a is this time known. When solving it one gets the following result:

$$p_2 = \frac{1}{16}x^4 - \frac{3x^3}{4} + 3x^2 - 4x + 2$$

4 Multi-variable Polynomial Interpolation

The polynomial interpolation can also be used with a Multi-variate Polynomial Interpolation. Where the polynomial is represented by Equation 6.

$$\begin{aligned} p(x,y) &= a_{0,0}\pi_0(x)\pi_0(y) + a_{1,0}\pi_1(x)\pi_0(y) + a_{0,1}\pi_0(x)\pi_1(y) + a_{1,1}\pi_1(x)\pi_1(y), \\ p(x,y) &= a_{0,0}1\cdot 1 + a_{1,0}(x-x_0)1 + a_{0,1}1\left(y-y_0\right) + a_{1,1}(x-x_0)\left(y-y_0\right) \end{aligned} \tag{6}$$

4.1 Example: Multi-variable polynomial Interpolation

$p(x,y)$ $p(0,0) = 0, p(1,0) = 1, p(0,1) = 0, p(1,1) = 0.5$ Now lets calculate the first x row:

$$\begin{array}{c|c|c} x & z & y' \\ \hline 0 & \underbrace{0}_{a_0} & \\ & & \frac{1-0}{1-0} = 1 \\ 1 & 1 & \end{array}$$

$$p(x; y_0 = 0) = 0 + 1\cdot(x-0)$$

Now lets calculate the second x row:

$$\begin{array}{c|c|c} x & z & y' \\ \hline 0 & \underbrace{1}_{a_0} & \\ & & \frac{0.5-1}{1-0} = -\frac{1}{2} \\ 1 & 0.5 & \end{array}$$

$$p(x; y_1 = 1) = 1 + -\frac{1}{2}\cdot(x-0)$$

And in step 3 we combine those two.

$$\begin{array}{c|c|c} y & z & y' \\ \hline 0 & \underbrace{x}_{a_0} & \\ & & \frac{(1-\frac{1}{2}x)-x}{1-0} = \underbrace{1-\frac{3}{2}x}_{a_1} \\ 1 & 1-\frac{1}{2}x & \end{array}$$

$$p(x,y) = x\cdot 1 + (1-\frac{3}{2}x)\cdot(y-0) = \underline{\underline{x - \frac{1}{2}\cdot x\cdot y}}$$

4.1.1 Alternative Method:

$$p(x,y) = a_{0,0}\pi_0(x)\pi_0(y) + a_{1,0}\pi_1(x)\pi_0(y) + a_{0,1}\pi_0(x)\pi_1(y) + a_{1,1}\pi_1(x)\pi_1(y)$$

$$\underbrace{p(0,0)}_{=0} = a_{0,0}\cdot 1 = 0 \Rightarrow a_{0,0} = 0$$

$$\underbrace{p(1,0)}_{=1} = a_{0,0}\cdot 1 + a_{1,0}\cdot \underbrace{x}_{=1} = 1 \Rightarrow a_{1,0} = 1$$

$$\underbrace{p(0,1)}_{=0} = a_{0,0}\cdot 1 + a_{0,1}\cdot \underbrace{y}_{=1} = 0 \Rightarrow a_{0,1} = 0$$

$$\underbrace{p(1,1)}_{=0.5} = \underbrace{a_{0,0}\cdot 1}_{=0} + \underbrace{a_{1,0}\cdot \underbrace{x}_{=1}}_{=1} + \underbrace{a_{0,1}\cdot \underbrace{y}_{=1}}_{=0} + a_{1,1}\cdot \underbrace{x}_{=1}\cdot \underbrace{y}_{=1} = 0 \Rightarrow a_{1,1} = -0.5$$

$$p(x,y) = \underline{\underline{1\cdot x - \frac{1}{2}x\cdot y}}$$

4.2 Questions

How many conditions are generally required

- for a tri-linear polynomial interpolation
  - We have three dimensions and a linear polynomial ⇒ we have 8 basis functions $(2\cdot 2\cdot 2)$ ⇒ 8 conditions that are required:

$$\{\pi_0(x);\pi_1(x)\} \rightarrow \{1;x\} = \mathrm{A}$$
$$\{\pi_0(y);\pi_1(y)\} \rightarrow \{1;y\} = \mathrm{B}$$
$$\{\pi_0(z);\pi_1(z)\} \rightarrow \{1;z\} = \mathrm{C}$$
$$\mathrm{A}\cdot\mathrm{B}\cdot\mathrm{C} \Rightarrow \text{the basis is} 1; x, y, z, xy, xz, yz, xyz$$

- for a tri-cubic polynomial interpolation
  - We have again three dimensions and each dimension has four newton basis polynomials ⇒ $4\cdot 4\cdot 4 = 64$
  - We would need to define;

$$\text{8 points}$$
$$\text{24 first derivatives, at each point three } \frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}$$
$$\text{24 second derivatives, at each point three } \frac{\delta^2}{\delta x\cdot\delta y}, \frac{\delta^2}{\delta x\cdot\delta z}, \frac{\delta^2}{\delta y\cdot\delta z}$$
$$\text{8 third derivatives, at each point one } \frac{\delta^3}{\delta x\cdot\delta y\cdot\delta z}$$

5 Spline Interpolation

5.1 Idea

The Idea of the spline interpolation is that one does not interpolate the data with a high degree polynomial, but with multiple polynomials of lower degree. Due to that the runge phenomenon does not occur which occurs for high deggre interplations. When following this approach the transition of one spline ot the next must be considered. Normally one says that the derivative up ot an order n must be the same from one to the next spline. The drawback of this approach is that one needs a lot of storage, since one needs to store a lot of fucntions. The advantage is that it is easier to calculate, since the newton interpolation has a complexity of n^2 whereas a cubic spline interpolation has a complexity of n.

5.2 Cubic Spline

5.2.1 Solve problem

A spline can be described with Equation 7

$$\begin{aligned} \mathrm{S}_i(x) &= a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3 \\ \mathrm{S}_i' &= b_i + 2c_i(x-x_i) + 3d_i(x-x_i)^2 \\ \mathrm{S}_i'' &= 2c_i + 6d_i(x-x_i) \end{aligned} \tag{7}$$

5.2.2 Natrual Splining

In natural splines the energy is minimized, therefore $\underline{y_0''=0} = y_n'' \left(\min \int \left|f''(x)\right|^2 \, \mathrm{d}x\right)$

- For natural spline c_0 and c_n are given by Equation 8

$$c_0 = c_n = 0 \tag{8}$$

- For cubic splines the a coefficients can be calculated according to Equation 9

$$a_i = y_i \quad (i = 0, \dots, n-1) \tag{9}$$

- The b coefficients can be calculated according to Equation 10

$$\begin{aligned} b_{n-1} &= \frac{y_n - a_{n-1}}{h_{n-1}} - c_{n-1}h_{n-1} - d_{n-1}h_{n-1}^2 = \frac{y_n - y_{n-1}}{h_{n-1}} - \frac{2}{3}c_{n-1}h_{n-1} \\ b_{n-1} &= b_{n-2} + 2c_{n-2}h_{n-2} + 3d_{n-2}{h_{n-2}}^2 = \frac{a_{n-1}-a_{n-2}}{h_{n-2}} - \frac{2c_{n-2}+c_{n-1}}{3}h_{n-2} + 2c_{n-2}h_{n-2} + (c_{n-1}-c_{n-2})\,h_{n-2} \end{aligned} \tag{10}$$

- The d coefficients are given by Equation 11

$$d_{n-1} = -\frac{c_{n-1}}{3h_{n-1}} \tag{11}$$

- solve equation system after c_i:

$$
\begin{bmatrix}
2(h_0+h_1) & h_1 & & & & \\
h_1 & 2(h_1+h_2) & h_2 & & & \\
 & h_2 & 2(h_2+h_3) & h_3 & & \\
 & \ddots & \ddots & \ddots & & \\
 & & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} \\
 & & & h_{n-2} & 2(h_{n-2}+h_{n-1})
\end{bmatrix}
\cdot
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix}
=
\begin{pmatrix}
3\left(\frac{y_2-y_1}{h_1}-\frac{y_1-y_0}{h_0}\right) \\
3\left(\frac{y_3-y_2}{h_2}-\frac{y_2-y_1}{h_1}\right) \\
3\left(\frac{y_4-y_3}{h_3}-\frac{y_3-y_2}{h_2}\right) \\
\vdots \\
3\left(\frac{y_{i+1}-y_i}{h_i}-\frac{y_i-y_{i-1}}{h_{i-1}}\right) \\
3\left(\frac{y_{i+1}-y_i}{h_i}-\frac{y_i-y_{i-1}}{h_{i-1}}\right)
\end{pmatrix}
\quad (12)
$$

Error Calculation for cubic splines with C^2

$$
\begin{aligned}
|y(x)-S(x)| &\le \max\frac{\left|y^{(4)}(x)\right|}{4!}\frac{5H^4}{16} = \max\left|y^{(4)}(x)\right|\frac{5}{384}H^4 \\
\left|y'(x)-S'(x)\right| &\le \max\frac{\left|y^{(4)}(x)\right|}{4!}H^3 = \max\left|y^{(4)}(x)\right|\frac{1}{24}H^3 \\
\left|y''(x)-S''(x)\right| &\le \max\left|y^{(4)}(x)\right|\frac{3}{8}H^2 \quad x\in[x_0,x_n], \quad H=\max_{i=0,\dots,n-1} h_i
\end{aligned}
\quad (13)
$$

5.2.3 Formulas for the cubic clamped spline interpolation S

- $b_0 = y_0'; b_n = y_n'$
- solve equation system after c_i:

$$
\begin{bmatrix}
2h_0 & h_0 & & & & \\
h_0 & 2(h_0+h_1) & h_1 & & & \\
 & h_1 & 2(h_1+h_2) & h_2 & & \\
 & & h_2 & 2(h_2+h_3) & h_3 & \\
 & & \ddots & \ddots & \ddots & \\
 & & & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} \\
 & & & & 2h_{n-2} & 4h_{n-2}+3h_{n-1}
\end{bmatrix}
\cdot
\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix}
=
\begin{pmatrix}
3\left(\frac{a_1-a_0}{h_0}-y_0'\right) \\
\vdots \\
\left(3\left(\frac{y_{i+1}-y_i}{h_i}-\frac{y_i-y_{i-1}}{h_{i-1}}\right)\right)_{i=1,\dots n-2} \\
\vdots \\
9\frac{y_n-a_{n-1}}{h_{n-1}}-6\frac{a_{n-1}-a_{n-2}}{h_{n-2}}-3y_n'
\end{pmatrix}
\quad (14)
$$

- $b_{i-1} = \frac{a_i-a_{i-1}}{h_{i-1}} - \frac{2c_{i-1}+c_i}{3}h_{i-1} \quad (i=1,\dots n-1)$
- $b_{n-1} = \frac{y_n-a_{n-1}}{h_{n-1}} - c_{n-1}h_{n-1} - d_{n-1}h_{n-1}^2 = \frac{y_n-y_{n-1}}{h_{n-1}} - \frac{2}{3}c_{n-1}h_{n-1}$
- $d_{i-1} = \frac{c_i-c_{i-1}}{3h_{i-1}} \quad (i=1,\dots n-1)$
- Clamped splining: $d_{n-1} = \frac{y_n'-b_{n-1}-2c_{n-1}h_{n-1}}{3h_{n-1}^2}$ Natural Splining: $d_{n-1} = -\frac{c_{n-1}}{3h_{n-1}}$

5.2.4 Example Natural Spline

Calculate the natural cubic spline-interpolation for a sine function $sin(x)$ in the interval $[0,\pi]$ according to the points $\{0, \frac{\pi}{2}, \pi\}$. Also calculate the maximum error.
From Equation 7 one knows that

$$
\begin{aligned}
S_0(x) &= a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3 \\
S_1(x) &= a_1 + b_1(x-\frac{\pi}{2}) + c_1(x-\frac{\pi}{2})^2 + d_1(x-\frac{\pi}{2})^3
\end{aligned}
$$

Furthermore $\underline{c_0 = 0}$ (Equation 8) since we use natural splines and $\underline{a_0 = y_0 = 0, a_1 = y_1 = 1}$ (Equation 9). From Equation 13 one can write down the following equations:

$$
\begin{aligned}
(2(h_0+h_1))\cdot(c_1) &= 3\left(\frac{y_2-y_1}{h_1} - \frac{y_1-y_0}{h_0}\right) \\
\left(2\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right)\cdot(c_1) &= 3\left(\frac{0-1}{\frac{\pi}{2}} - \frac{1-0}{\frac{\pi}{2}}\right) \\
(2\pi) = 3\left(\frac{-12}{\pi}\right) &\Rightarrow \underline{c_1 = \frac{-6}{\pi^2}}
\end{aligned}
$$

From Equation 10 one knows that

$$b_0 = \frac{y_1-y_0}{\pi/2} - \frac{2c_0-c_1}{3}\cdot\frac{\pi}{2} = \frac{2}{\pi}+\frac{2}{\pi^2}+\frac{\pi}{2} = \underline{\frac{3}{\pi}}$$

From Equation 11 one knows that

$$d_0 = \frac{c_1-c_0}{3\cdot\frac{\pi}{2}} = \frac{-6}{\pi^2}\cdot\frac{2}{3\cdot\pi} = \underline{\frac{-4}{\pi^3}}$$

$$d_1 = \frac{c_2-c_1}{3\cdot\frac{\pi}{2}} = \underline{\frac{4}{\pi^3}}$$

And finally from Equation 10 that:

$$b_1 = \frac{y_2-y_1}{\pi/2} - \frac{2c_1-c_2}{3}\cdot\frac{\pi}{2} = \frac{2}{\pi}+\frac{2}{\pi} = \underline{0}$$

The error can then be estimated with Equation 13.

$$|y-s| \le \max_{[0,\pi]} \frac{\overbrace{|y^4(\xi)|}^{sin(\xi)}}{4!}\cdot\frac{5\cdot\left(\frac{\pi}{2}\right)^4}{16} = 1\cdot\underline{\frac{5\cdot\pi^4}{384\cdot 16}}$$

5.2.5 Example

The data below was generated by the sine function. In this example, the natural and clamped spline as well as the max error are calculated.

x_i	0	$\pi/3$	$2\pi/3$	π
y_i	0	$\sqrt{3}/2$	$\sqrt{3}/2$	0

natural Spline

$$\begin{pmatrix} 2\frac{2\pi}{3} & \frac{\pi}{3} \\ \frac{\pi}{3} & 2\frac{2\pi}{3} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3\left(\frac{a_2-a_1}{h} - \frac{a_1-a_0}{h}\right) \\ 3\left(\frac{a_3-a_2}{h} - \frac{a_2-a_1}{h}\right) \end{pmatrix} = \begin{pmatrix} -\frac{9\sqrt{3}}{2\pi} \\ -\frac{9\sqrt{3}}{2\pi} \end{pmatrix} \tag{15}$$

$$\Longrightarrow c_1 = \frac{-27\sqrt{3}}{10\pi^2}; c_2 = \frac{-27\sqrt{3}}{10\pi^2}$$

$$\Longrightarrow d_0 = \frac{c_1 - c_0}{3\left(\frac{\pi}{3}\right)} = \frac{-27\sqrt{3}}{10\pi^3}; d_1 = \frac{c_2 - c_1}{3 \cdot h} = 0 \text{ (??)}$$

$$\Longrightarrow d_2 = \frac{-c_2}{3h} = \frac{27\sqrt{3}}{\pi^3 10} \text{ (??)}$$

$$\Longrightarrow b_0 = \frac{a_1 - a_0}{h} - \frac{2c_0 + c_1}{3} \cdot h = \frac{9\sqrt{3}}{5\pi} \text{ (??)}$$

$$\Longrightarrow b_1 = \frac{a_2 - a_1}{h} - \frac{2c_1 + c_2}{3} h = \frac{9\sqrt{3}}{10\pi} \text{ (??)}$$

$$\Longrightarrow b_2 = \frac{a_3 - a_2}{h} - c_2 \frac{\pi}{3} - d_2 \left(\frac{\pi}{3}\right)^2 = \frac{-9\sqrt{3}}{10\pi} \text{ (??)}$$

$$\mathrm{S}_0(x) = 0 + \frac{9\sqrt{3}}{5\pi}(x-0) + 0(x-0)^2 - \frac{27\sqrt{3}}{10\pi^3}(x-0)^3 \text{ (Equation 7)}$$

$$\left(0 \le x \le \frac{\pi}{3}\right)$$

$$\mathrm{S}_1(x) = \frac{\sqrt{3}}{2} + \frac{9\sqrt{3}}{10\pi}\left(x - \frac{\pi}{3}\right) - \frac{27\sqrt{3}}{10\pi^2}\left(x - \frac{\pi}{3}\right)^2 + 0\left(x - \frac{\pi}{3}\right)^3$$

$$\left(\frac{\pi}{3} \le x \le \frac{2\pi}{3}\right)$$

$$\rho_2(x) = \frac{\sqrt{3}}{2} - \frac{9\sqrt{3}}{10\pi}\left(x - \frac{2\pi}{3}\right) - \frac{27\sqrt{3}}{10\pi^2}\left(x - \frac{2\pi}{3}\right)^2 + \frac{27\sqrt{3}}{10\pi^3}\left(x - \frac{2\pi}{3}\right)^3$$

$$\left(\frac{2\pi}{3} \le x \le \pi\right)$$

clamped Spline

$$\begin{pmatrix} 2 \cdot \frac{\pi}{3} & \frac{\pi}{3} & \\ \frac{\pi}{3} & 2 \cdot \left(\frac{\pi}{3} + \frac{\pi}{3}\right) & \frac{\pi}{3} \\ & 2 \cdot \frac{\pi}{3} & 4\frac{\pi}{3} + 3\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3\left(\frac{y_1 - y_0}{h} - y_0'\right) \\ 3\left(\frac{y_2 - y_1}{h} - \frac{y_1 - y_0}{h}\right) \\ 9\left(\frac{y_3 - y_2}{h}\right) - 6\left(\frac{y_2 - y_1}{h}\right) - 3y_3' \end{pmatrix} = \begin{pmatrix} \frac{9\sqrt{3}}{2\pi} - 3 \\ -\frac{9\sqrt{3}}{2\pi} \\ 9\left(\frac{-\sqrt{3} \cdot 3}{2\pi}\right) - 6 \cdot 0 + 3 = \frac{-27\sqrt{3}}{2\pi} + 3 \end{pmatrix} \quad (16)$$

$$\Longrightarrow c_0 = \frac{-10\pi + 18\sqrt{3}}{2\pi^2}; c_1 = \frac{2\pi - 9\sqrt{3}}{2\pi^2}; \quad c_2 = \frac{2\pi - 9\sqrt{3}}{2\pi^2}$$

$$\Longrightarrow d_0 = \frac{c_1 - c_0}{3\hbar} = \frac{1}{\pi}\frac{12\pi - 27\sqrt{3}}{2\pi^2}; d_1 = \frac{c_2 - c_1}{3h} = \frac{1}{\pi} \cdot 0 = 0 \text{ (??)}$$

$$\Longrightarrow d_2 = \left(\frac{y_3 - y_2}{h} - c_2 h - b_2\right) / h^2 = \ldots = \frac{27\sqrt{3} - 12\pi}{2\pi 3} \text{ (?)}$$

$$\Longrightarrow b_0 = \frac{y_1 - y_0}{h} - \frac{2c_0 + c_1}{3} h = \frac{3}{\pi}\frac{\sqrt{3}}{2} - \frac{-18\pi + 27\sqrt{3}}{2\pi^2 \cdot 3} \cdot \frac{\pi}{3} = \frac{6\pi - 0\sqrt{3}}{2\pi \cdot 3} = 1 \text{ (??)}$$

$$\Longrightarrow b_1 = \frac{y_2 - y_1}{h} - \frac{2c_1 + c_2}{3} h = 0 - \frac{6\pi - 27\sqrt{3}}{2\pi^2 \cdot 3} \cdot \frac{\pi}{3} = -\frac{2\pi - 9\sqrt{3}}{6\pi} \text{ (??)}$$

$$\Longrightarrow b_2 = b_1 + 2c_1 h + 3d_1 h^2 = \ldots = \frac{1}{3} - \frac{3\sqrt{3}}{2\pi} \text{ (??)}$$

For the two examples above (natural and clamped) give maximum estimations for the following error quantities $|\, y(x) - \mathrm{S}(x) \,|$, $|\, y'(x) - \mathrm{S}'(x) \,|$ and $|\, y''(x) - \mathrm{S}''(x) \,|$. The osculation error of a cubic spline can be calculated with **??** and the osculation error of a periodic cubic spline ($y_0 = y_n \Rightarrow \mathrm{S}(x_0) = \mathrm{S}(x_n)$) with Equation 13.,

$$
\begin{aligned}
| y - s | &\le \max | y^{(4)}(x) | \cdot \frac{5}{384} \mathrm{H}^4 \qquad (\mathrm{H} = \max h_i) \\
&= \max | \sin(x) | \cdot \frac{5}{384} \cdot \left(\frac{\pi}{3}\right)^4 \\
&= 1 \cdot \frac{5}{384} \cdot \frac{\pi^4}{81} \approx 0.01565
\end{aligned}
$$

$$
\left| y' - \mathrm{S}' \right| \le \max | y^{(4)}(x) | \frac{\mathrm{H}^3}{24} = 1 \cdot \frac{\pi}{27 \cdot 24} \approx 0.047849
$$

$$
\left| y'' - \mathrm{S}'' \right| \leqslant \max | y^{(4)}(x) | \frac{3}{8} \mathrm{H}^2 = 1 \cdot \frac{3}{8} \frac{\pi^2}{9} \approx 0.411234
$$

5.3 Bernstein-Bézier Splines (B-B-Splines)

The bernstein-Bézier splines should give the same result as the cubic splines mentioned in the previous chapter. The difference is that one does not get a single formula in the end, but different data points. A good explanation can be found in the following video. But first of all to understand bézier curves/spline one must be familiar with Bernstein polynomials and therefore with the binomial coefficient (see also Equation 17)

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{17}$$

$$\begin{array}{ll}
\text{Row 0:} & \binom{0}{0} \\
\text{Row 1:} & \binom{1}{0} \quad \binom{1}{1} \\
\text{Row 2:} & \binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\text{Row 3:} & \binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\text{Row 4:} & \binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\text{Row 5:} & \binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\text{Row 6:} & \binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\vdots & \quad \iddots \qquad\qquad\qquad\qquad \ddots \\
\text{Row } 2n\text{:} & \binom{2n}{0} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad \binom{2n}{2n}
\end{array}$$

Figure 1: Pascal's triangle formula

Example: (For what can the binomial coefficients be used?) What is the fourth term of $(3x-4y)^6$ (note there is no zeroth term, therefore we subtract one in the equation blew)? The result can also be read from Figure 2.

$$\binom{6}{4-1} = \frac{6!}{(4-1)!(6-(4-1))!} = 20$$

Therefore, the fourth term is:

$$20 \cdot \left((3x)^{6-(4-1)} \cdot (-4y)^{(4-1)}\right) = 20 \cdot \left(27 \cdot x^3 \cdot (-64)\, y^4\right) = -34560x^2y^4$$

This is much easier than to really calculate the polynomial. An explanation can also be found in the following video

5.3.1 Bernstein Polynomial

The bernstein polynomial is defined in Equation 18.

$$\mathrm{B_{in}}\,(t) = \binom{n}{i}(1-t)^{n-i}t^i \quad t \in [0,1] \quad (i = 0, 1, \cdots, n) \tag{18}$$

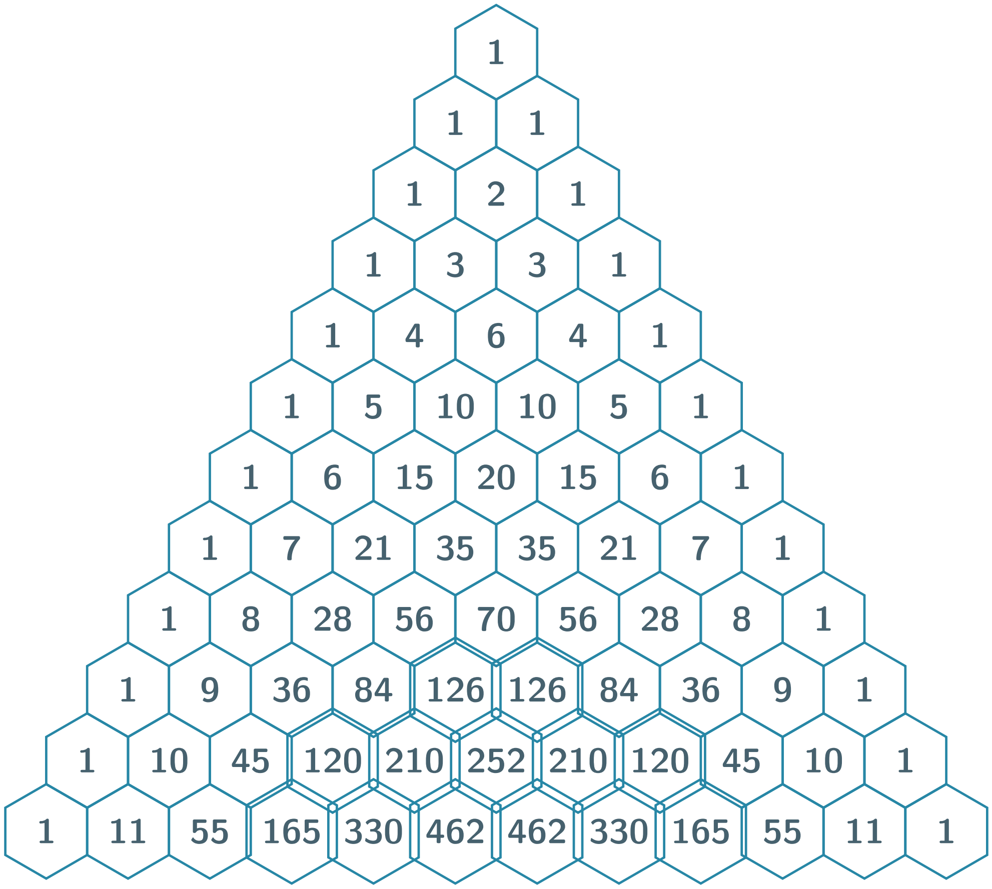

Figure 2: Pascal's triangle numbers

In Equation 18 one had an interval $\in [0,1]$. When our data is in another range like $\in [a,b]$ one has to use Equation 19

$$\mathrm{B}_{in}(u,a,b) = \mathrm{B}_{\mathrm{in}}\left(\frac{u-a}{b-a}\right) = \frac{1}{(b-a)^n}\begin{pmatrix} n \\ i \end{pmatrix}(b-u)^{n-i}(u-a)^i \quad u \in [a,b] \quad (i = 0,1,\cdots,n) \tag{19}$$

Furthermore note that those polynomials look like one can see in Figure 3. The polynomials relate to each other, as one can see in Equation 20.

$$\begin{aligned}
&\frac{d}{dt}\mathrm{B}_{i,n}(t) = n\left(\mathrm{B}_{i-1,n-1}(t) - \mathrm{B}_{i,n-1}(t)\right) = -n\Delta \mathrm{B}_{i-1,n-1}(t) \\
&\frac{d^2}{dt^2}\mathrm{B}_{i,n}(t) = n(n-1)\left(\mathrm{B}_{i-2,n-2}(t) - 2\mathrm{B}_{i-1,n-2}(t) + \mathrm{B}_{i,n-2}(t)\right) = n(n-1)\Delta^2 \mathrm{B}_{i-2,n-2}(t) \\
&\frac{d^k}{dt^k}\mathrm{B}_{i,n}(t) = (-1)^k n(n-1)\dots(n-k+1)\Delta^k \mathrm{B}_{i-k,n-k}(t)
\end{aligned} \tag{20}$$

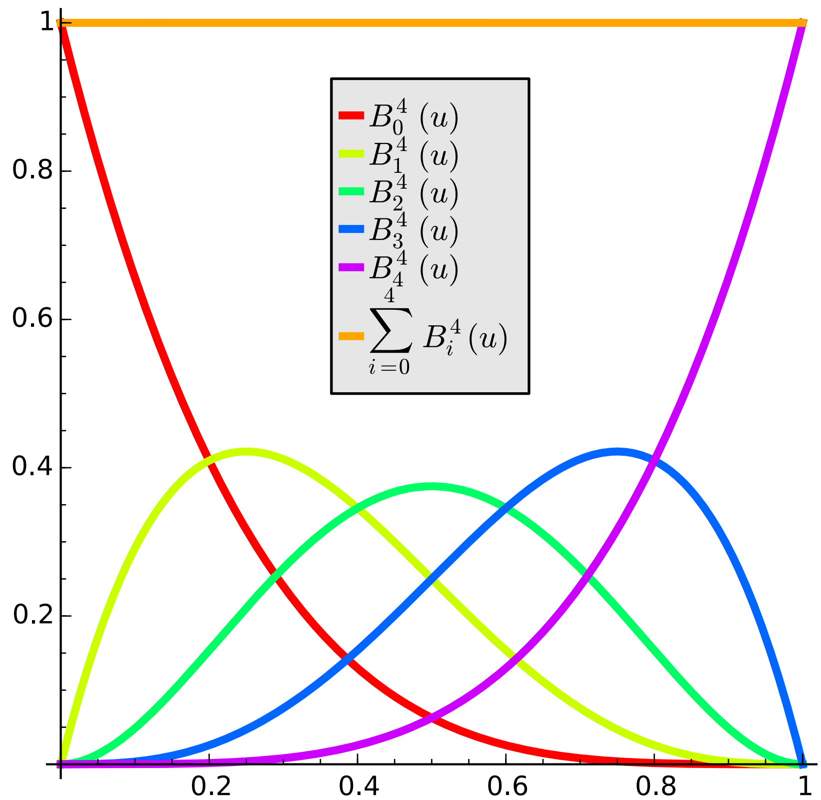

Figure 3: Plot of Bernstein polynomial functions up to degree 4 with summation of all four functions to show characteristic of partition of one (Note the maximum of each polynomial is always at $t = \frac{i}{n}$)

5.3.2 Simple Bézier Curves

A simple Bézier curve is defined with Equation 21. To get the idea also have a look at Figure 4.

$$\vec{r}(t) = \sum_{i=0}^{n} \vec{\mathrm{P}}_i \mathrm{B}_{in}(t) \quad t \in [0,1] \tag{21}$$

Where:

- i control point number
- n is the total number of control points minus one, since Points are $(\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2, \mathrm{P}_n)$

5.3.3 Composite Bézier Curves

The simple Bézier curves meet at common control points. Which is a continuity condition (C^0), but often higher (smoothness) conditions are required (C^k-smooth). This condition is only met if and only if Equation 22 is given.

$$\frac{\Delta^\ell \vec{\mathrm{P}}_{n-\ell,j}}{h_j^\ell} = \frac{\Delta^\ell \vec{\mathrm{P}}_{0,j+1}}{h_{j+1}^\ell} \quad (j = 0, 1, \cdots, m-2 \quad \ell = 0, 1, 2, \cdots, k) \tag{22}$$

Writing out Equation 22 for C^1 smoothness results in Equation 23, whereas C^2 smoothness results in Equation 24 where one has to know that also Equation 23 C^1 smoothness must be met.

$$\frac{n\left(\vec{\mathrm{P}}_{n,j} - \vec{\mathrm{P}}_{n-1,j}\right)}{h_j} = \frac{n\left(\vec{\mathrm{P}}_{1,j+1} - \vec{\mathrm{P}}_{0,j+1}\right)}{h_{j+1}} \quad (j = 0, 1, \cdots, m-2) \tag{23}$$

$$\frac{n(n-1)\left(\vec{P}_{n,j}-2\vec{P}_{n-1,j}+\vec{P}_{n-2,j}\right)}{h_j^{\,2}}=\frac{n(n-1)\left(\vec{P}_{2,j+1}-2\vec{P}_{1,j+1}+\vec{P}_{0,j+1}\right)}{h_{j+1}^{\,2}}\quad (j=0,1,\cdots,m-2) \tag{24}$$

The Bézier-curve is defined by the control points $\left(\vec{P}_0,\vec{P}_1,\dots\vec{P}_n(n\geq 2)\right)$ and the Bernstein polynomials. On each spline one has n+1 control points.

- spline number, the spline number there are m splines (m=(Number of given Data points-1))
- point on spline, there are $n+1$ points on the spline

$$\vec{r}_j(u)=\sum_{i=0}^{n}\vec{P}_{i,j}B_{in}\left(u,u_j\cdot u_{j+1}\right)\quad u\in\left[u_j\cdot u_{j+1}\right]\quad (j=0,1,\cdots,m-1)$$

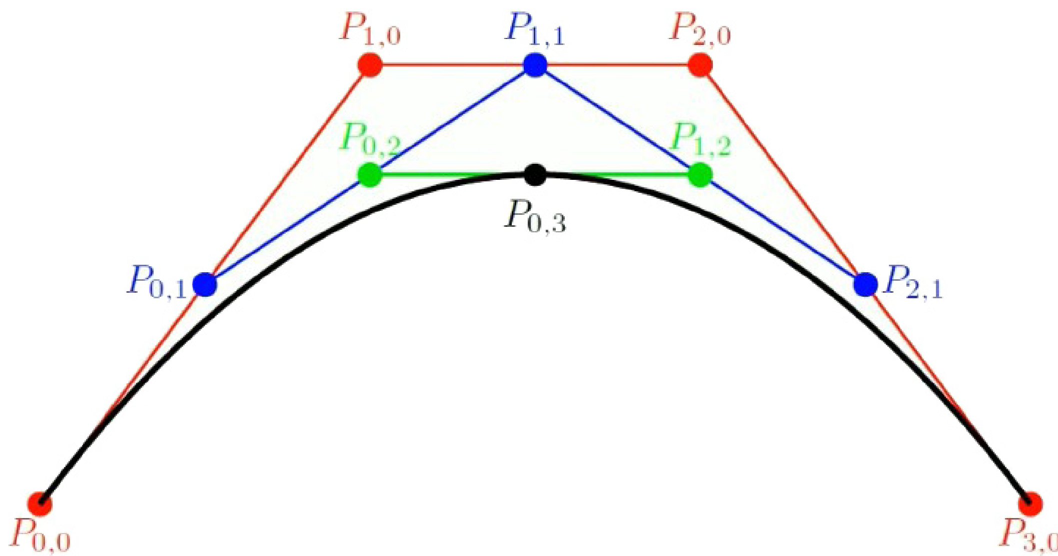

Figure 4: Cubic Bézier Curve (Always defined by 4 control points), degree three

5.3.4 Example: Composite Bézier Curves

The four points $A=(0,0), B=(1,0), C=(2,3)$ and $D=(2,4)$ are to be interpolated (joined) by composed C^1 Bernstein-Bézier splines: A and B are to be joined linearly (by a straight line), as well as C and D.
Compute the missing C^1 Bernstein-Bézier spline of minimal degree between B and C.

To solve the exercise one can use Equation 23. Where the first spline and the last one have two control points, since it is a straight line $\Rightarrow n=1$. The second spline has four conditions, since tow points must be met and two derivatives, since it must be C^1 smooth. Due to that $n=3\Rightarrow$ the degree is also 3 (also called cubic).

$$1\cdot\left(P_{1,0}-P_{0,0}\right)=3\cdot\left(P_{1,1}-P_{0,1}\right)=1\cdot\left(\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\0\end{pmatrix}\right)=3\cdot\left(P_{1,1}-\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$3\cdot\left(P_{3,1}-P_{2,1}\right)=1\cdot\left(P_{1,2}-P_{0,2}\right)=3\cdot\left(\begin{pmatrix}2\\3\end{pmatrix}-P_{2,1}\right)=1\cdot\left(\begin{pmatrix}0\\0\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$\Rightarrow P_{1,1}=\begin{pmatrix}\frac{4}{3}\\0\end{pmatrix}$$

$$\Rightarrow P_{2,1}=\begin{pmatrix}2\\\frac{8}{3}\end{pmatrix}$$

$$\vec{r}(t) = \underbrace{P_{0,1}}_{B} \cdot B_{03}(t) + P_{1,1} \cdot B_{13}(t) + P_{2,1} \cdot B_{23}(A) + \underbrace{P_{0,1}}_{C} \cdot B_{33}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1-t)^3 + 3 \begin{pmatrix} \frac{4}{3} \\ 0 \end{pmatrix} (1-t)^2 t + 3 \begin{pmatrix} 2 \\ \frac{8}{3} \end{pmatrix} (1-t) t^2 + \begin{pmatrix} 2 \\ 3 \end{pmatrix} t^3$$

$$t \in [0,1]$$

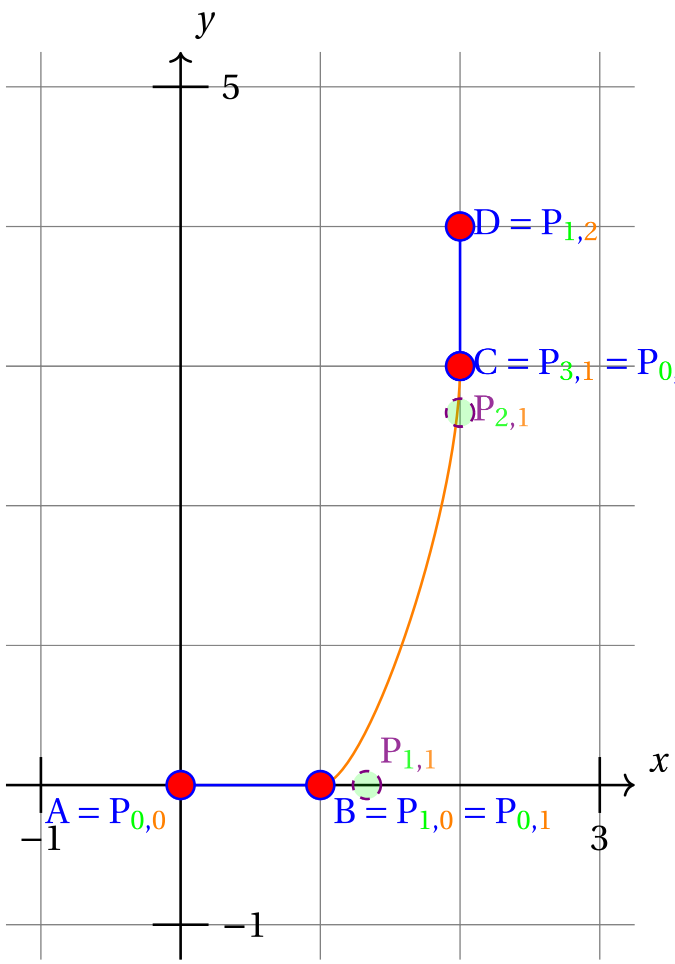

Figure 5: Exercise Overview

5.3.5 Properties

- The Bézier-curves is always inside the convex hull of the data points.
-

$$\vec{r}(0) = \vec{P}_0 \quad \vec{r}(1) = \vec{P}_n$$
$$\vec{r}\,'(0) = n\left(\vec{P}_1 - \vec{P}_0\right) \quad \vec{r}\,'(1) = n\left(\vec{P}_n - \vec{P}_{n-1}\right)$$
$$\vec{r}\,''(0) = n(n-1)\left(\vec{P}_2 - 2\vec{P}_1 + \vec{P}_0\right) \quad \vec{r}\,''(1) = n(n-1)\left(\vec{P}_n - 2\vec{P}_{n-1} + \vec{P}_{n-2}\right)$$

- If C^k smoothness for a point is required, k equations or k control points are necessary

5.3.6 Casteljau recurrence

The Casteljau recurrence is a similar idea as the neville-aitken. With this Idea a point on the Bézier curve can be calculated as a linear combination of two points on a Bézier curve of a lower degree.

$$\vec{r}_{\overrightarrow{P_0},\overrightarrow{P_1},\dots,\overrightarrow{P_n}}(t) = (1-t)\cdot\vec{r}_{\overrightarrow{P_0},\overrightarrow{P_1},\dots,P_{n-1}\rightarrow}(t) + t\cdot\vec{r}_{\overrightarrow{P_1},\overrightarrow{P_2},\dots,\overrightarrow{P_n}}(t) \quad t\in[0,1]$$

$$\begin{array}{llll}
C^0: & & P_n = \vec{Q}_0 & \\
C^1: & \vec{r}_P'(1) = & n\left(\vec{P}_n - \vec{P}_{n-1}\right) = m\left(\vec{Q}_1 - \vec{Q}_0\right) & = \vec{r}_Q'(0) \\
C^2: & \vec{r}_P''(1) = & n(n-1)\left(\vec{P}_n - 2\vec{P}_{n-1} + \vec{P}_{n-2}\right) = m(m-1)\left(\vec{Q}_2 - 2\vec{Q}_1 + \vec{Q}_0\right) & = \vec{r}_Q''(0) \\
C^k: & \vec{r}_P^{(k)}(1) = & n(n-1)\dots(n-k+1)\left(\Delta^k\vec{P}_{n-k}\right) = m(m-1)\dots(m-k+1)\left(\Delta^k\vec{Q}_0\right) & = \vec{r}_Q^{(k)}(0)
\end{array}$$

Is $\overrightarrow{r_j}(t) = \sum_{i=0}^{n}\vec{P}_{i,j}B_{i,n}(t) \quad t\in[0,1]$ with the functions Q_j defined (degree: n), the following formulas turn out:

$$\begin{array}{llll}
C^0: & \vec{P}_{n,j-1} = \vec{P}_{0,j} = \vec{Q}_j & & \\
C^1: & \vec{r}_{j-1}'(1) = & \vec{Q}_j - \vec{P}_{n-1,j-1} = \vec{P}_{1,j} - \vec{Q}_j & = \vec{r}_j'(0) \\
C^2: & \vec{r}_{j-1}''(1) = & \vec{Q}_j - 2\vec{P}_{n-1,j-1} + \vec{P}_{n-2,j-1} = \vec{P}_{2,j} - 2\vec{P}_{1,j} + \vec{Q}_j & = \vec{r}_j''(0) \\
C^k: & \vec{r}_{j-1}^{(k)}(1) = & \Delta^k\vec{P}_{n-k,j-1} = \Delta^k\vec{P}_{0,j} & = \vec{r}_j^{(k)}(0)
\end{array}$$

5.3.7 Example

Let's do the same example as in subsubsection 5.2.5. Where the following four points are given:

$$\left\{\underbrace{(0,0)}_{Q_0=P_{0,0}}, \underbrace{\left(\frac{\pi}{3},\frac{\sqrt{3}}{2}\right)}_{Q_1=P_{0,1}=P_{3,0}}, \left(\frac{2\pi}{3},\frac{\sqrt{3}}{2}\right), (\pi,0)\right\}$$

Therefore $\vec{Q}_0 = (0,0), \vec{Q}_1 = \left(\frac{\pi}{3},\frac{\sqrt{3}}{2}\right),\dots$ and $h_j = h = \frac{\pi}{3}$. One now has to met the following requirements:

$$\begin{aligned}
\vec{Q}_1 - \vec{P}_{2,0} &= \vec{P}_{1,1} - \vec{Q}_1 \\
\vec{Q}_2 - \vec{P}_{2,1} &= \vec{P}_{1,2} - \vec{Q}_2 \\
\vec{Q}_1 - 2\vec{P}_{2,0} + \vec{P}_{1,0} &= \vec{P}_{2,1} - 2\vec{P}_{1,1} + \vec{Q}_1 \\
\vec{Q}_2 - 2\vec{P}_{2,1} + \vec{P}_{1,1} &= \vec{P}_{2,2} - 2\vec{P}_{1,2} + \vec{Q}_2
\end{aligned}$$

$$\begin{aligned}
\vec{P}_{2,0} - 2\vec{P}_{1,0} + \vec{Q}_0 &= \overrightarrow{0} \\
\vec{Q}_3 - 2\vec{P}_{2,2} + \vec{P}_{1,2} &= \overrightarrow{0}
\end{aligned}$$

The solution of the linear system above gives us the following points:

$$\left\{\underbrace{\left\{\frac{\pi}{9},\frac{\sqrt{3}}{5}\right\}}_{P_{1,0}}, \underbrace{\left\{\frac{2\pi}{9},\frac{2\sqrt{3}}{5}\right\}}_{P_{2,0}}, \underbrace{\left\{\frac{4\pi}{9},\frac{3\sqrt{3}}{5}\right\}}_{P_{1,1}}, \underbrace{\left\{\frac{5\pi}{9},\frac{3\sqrt{3}}{5}\right\}}_{P_{2,1}}, \left\{\frac{7\pi}{9},\frac{2\sqrt{3}}{5}\right\}, \left\{\frac{8\pi}{9},\frac{\sqrt{3}}{5}\right\}\right\}$$

When we now calculate the first spline we get the same result as before.

$$\begin{aligned}
\vec{r}_1(t) &= \begin{pmatrix} \frac{1}{9}\pi B_{1,3}(t) + \frac{2}{9}\pi B_{2,3}(t) + \frac{1}{3}\pi B_{3,3}(t) \\ \frac{1}{5}\sqrt{3}\, B_{1,3}(t) + \frac{2}{5}\sqrt{3}\, B_{2,3}(t) + \frac{1}{2}\sqrt{3}\, B_{3,3}(t) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\pi}{9}3(1-t)^2 t + \frac{2\pi}{9}3(1-t)t^2 + \frac{1}{3}\pi t^3 \\ \frac{\sqrt{3}}{5}3(1-t)^2 t + \frac{2}{5}\sqrt{3}3(1-t)t^2 + \frac{\sqrt{3}}{2}t^3 \end{pmatrix} \\
&= \begin{pmatrix} x \\ y \end{pmatrix} \\
\Longleftrightarrow & \begin{pmatrix} \frac{\pi}{3}t = x \\ \frac{3\sqrt{3}}{5}t - \frac{\sqrt{3}}{10}t^3 = y \end{pmatrix} \Rightarrow \begin{pmatrix} t = \frac{3x}{\pi} \\ y = \frac{9\sqrt{3}}{5\pi}x - \frac{27\sqrt{3}}{10\pi^3}x \end{pmatrix}
\end{aligned} \tag{25}$$

For the second spline we get the following:

$$\begin{pmatrix} \frac{1}{3}\pi B_{0,3}(t) + \frac{4}{9}\pi B_{1,3}(t) + \frac{5}{9}\pi B_{2,3}(t) + \frac{2}{3}\pi B_{3,3}(t) \\ \frac{1}{2}\sqrt{3}\, B_{0,3}(t) + \frac{3}{5}\sqrt{3}\, B_{1,3}(t) + \frac{3}{5}\sqrt{3}\, B_{2,3}(t) + \frac{1}{2}\sqrt{3}\, B_{3,3}(t) \end{pmatrix} \tag{26}$$

6 Linear Least-Squares approximation

6.1 Idea

Interpolation with the collocation methods often run into oscillation problems for (rather large) sets of measurement points. Furthermore, in most cases the measurements also contain some error points, which one does not want to represent in the graph. Due to that, an approximation (data points are not represented exactly any more) might be the preferred way to represent the data.

6.2 Linear Least-Squares

To find the best approximation, one must define what is a good and what is a bad approximation, which could be with the following basic functions:

$$\Rightarrow \text{Basics of functions} \Rightarrow min(max\,|\,r_i\,|) \\ \Rightarrow \sum_{i\ldots} |\,r_i\,| \\ \Rightarrow \sum_{i\ldots} r_i^2$$

Note: Error = residuals ⇒ norm of residuals
Mathematically, the minimization of the squared errors is the easiest, therefore this one is most commonly used (least square approximation)

The approximation function can be described with a set of basis functions, which we name here $g_0, g_1, \ldots, g_m = \{g_j\}_{j=0,\ldots,m}$. Note the basis functions are sometimes also called monomials. Furthermore we define the following

- N: Number of sample points
- m: degree of basic functions (m ≪ N)
- a: weighting coefficients
- $(x_0, y_0), (x_1, y_1)_1, \ldots, (x_N, y_N) = \{(x_i, y_i)\}_{i=0,\ldots,N}$: measurement points

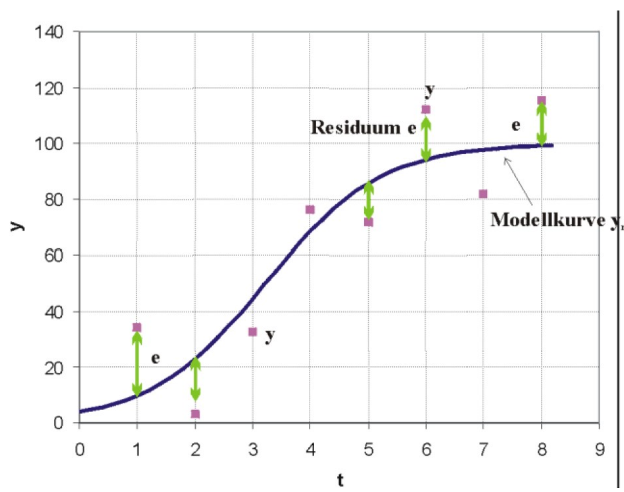

Figure 6: Data approximated by an error curve

With those variables one can create a equation in matrix notation form, as can be seen in Equation 27:

$$\overbrace{\begin{pmatrix} g_0(x_0) & g_1(x_0) & \ldots & g_m(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ g_0(x_N) & g_1(x_N) & \ldots & g_m(x_N) \end{pmatrix}}^{\text{Designmatrix G}} \cdot \begin{pmatrix} a_0 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ \vdots \\ y_N \end{pmatrix} \Leftrightarrow \quad G \cdot a = y \tag{27}$$

Since Equation 27 is normally overdetermined (m≪N) The error/residuals can be calculated with Equation 28 and the squared sum S of residuals with Equation 29. The goal is now to minimize S from Equation 29. This can be done with Equation 30, which is not derived in this post (for more information, search after orthogonal projection).

$$r_i = y_i - \sum_{j=0}^{m} a_j g_j(x_i) \quad (i = 0, \dots, \mathrm{N}) \tag{28}$$

$$\underbrace{\mathrm{S}}_{\text{Error}} = \sum_{i=0}^{\mathrm{N}} \left(y_i - \underbrace{\sum_{j=0}^{m} a_j g_j(x_i)}_{\text{Model}} \right)^2 = \sum_{i=0}^{\mathrm{N}} r_i^2 \Rightarrow \text{min!} \tag{29}$$

6.2.1 Thinking hint

Lets assume one has the following points: $\{1,1\},\{2,2\}$ and one wants to approximate those points by a polynomial of degree zero $(m = 0) \Rightarrow g_0(x) = 1$. Then one can write the term inside the square brackets as the following:

$$\overbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}^{y} - \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{\mathbf{G}} \cdot \overbrace{\begin{bmatrix} a \end{bmatrix}}^{a}$$

But as one can see one can not take the square of this term, therefore one has to multiply on both left sides with $\mathbf{G}^{\mathrm{T}}$ which has no effect on the end result, since one is only interested in the minimum.

$$\overbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}^{\mathrm{G^T}} \cdot \overbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}^{y} - \overbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}^{\mathrm{G^T}} \cdot \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{\mathbf{G}} \cdot \overbrace{\begin{bmatrix} a \end{bmatrix}}^{a}$$

When one squares the term above and says that $\mathbf{G}^{\mathrm{T}} \cdot y$ and $\mathbf{G}^{\mathrm{T}} \cdot \mathbf{G}$ belong to each other, (are not separable) and then takes furthermore the derivative, one gets the following result: $2 \cdot \left(a \cdot (\mathbf{G}^{\mathrm{T}} \cdot \mathbf{G}) - (\mathbf{G}^{\mathrm{T}} \cdot y)\right) \cdot (\mathbf{G}^{\mathrm{T}} \cdot \mathbf{G})$. When one sets this term to zero since one is interested in the minimum and solves it after a one gets the result from Equation 30.

6.2.2 Normal equations

$$\underbrace{\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}}_{\text{Normal matrix}} \cdot \boldsymbol{a} = \boldsymbol{G}^{\mathrm{T}}\boldsymbol{y} \quad \Rightarrow \quad \boldsymbol{a} = \left(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}\right)^{-1}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{y} \tag{30}$$

$$\underbrace{\underbrace{\boldsymbol{G}^{\mathrm{T}}}_{(m+1)\times(\mathrm{N}+1)} \cdot \underbrace{\boldsymbol{G}}_{(\mathrm{N}+1)\times(m+1)}}_{(m+1)\times(m+1)} \cdot \underbrace{\boldsymbol{a}}_{(m+1)\times 1} = \underbrace{\boldsymbol{G}^{\mathrm{T}}}_{(m+1)\times 1} \cdot \underbrace{\boldsymbol{y}}_{(\mathrm{N}+1)\times 1}$$

6.3 Singular-value decomposition (SVD)

A singular-value decomposition is one of the most widely used matrix operations in applied linear algebra.

6.3.1 Idea

Every Matrix $\boldsymbol{G}$ with the dimensions $(\mathrm{N}+1) \times (m+1)$ can be decomposed as the triple product $\boldsymbol{UDV}^{\mathrm{T}}$ whereas $\boldsymbol{U}$ is an orthogonal (N +1) × (N +1) -matrix, $\boldsymbol{D}$ is a (N +1) × (m +1) diagonal matrix and $\boldsymbol{V}$ again is orthogonal with dimensions (m +1) × (m +1). When a matrix is orthogonal, the following applies: $Q^{T}Q = QQ^{T} = I$ and $Q^{T} = Q^{-1}$. Due to that nice property, Equation 27 can be calculated according to Equation 32.

$$\boldsymbol{G} = \boldsymbol{UDV}^{\mathrm{T}} = \boldsymbol{U} \cdot \begin{bmatrix} d_{00} & 0 & \dots & 0 \\ 0 & d_{11} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & d_{mm} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} \cdot \boldsymbol{V}^{\mathrm{T}} \tag{31}$$

$Ga_{|} = y_{|}$ becomes $UDV^{T} a_{|} = y_{|}$. Which results in Equation 32

$$a_{|} = VD^{-1}U^{T} \cdot y_{|} \tag{32}$$

6.3.2 Uniform arguments and orthogonal polynomials

With uniform arguments $x_i - x_j = (j-i)h$ for all $i, j \rightarrow \{x_0 \dots x_N\} = \{x_0 + t \cdot h\}_{t=0 \dots N}$ and orthogonal polynomials GG^{T} can be diagonalized and therefore the equation can be easier solved. An example can be found in subsubsection 6.3.6

$$\begin{aligned} p_{k,\mathrm{N}}(t) &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{k+i}{i} \frac{t^{(i)}}{\mathrm{N}^{(i)}} \\ &= 1 + \sum_{i=1}^{k} (-1)^i \binom{k}{i} \binom{k+i}{i} \frac{t(t-1)(t-2)\dots(t-i+1)}{\mathrm{N}(\mathrm{N}-1)(\mathrm{N}-2)\dots(\mathrm{N}-i+1)} \quad (k = 1, \dots, \mathrm{N}) \end{aligned} \tag{33}$$

with $t = \frac{x-x_0}{h}$ $\quad \binom{k}{i} = \frac{k!}{i!(k-i)!} = n\mathrm{C}r(k,i)$

$p_{k,\mathrm{N}}$ can now with g_k be put in the design matrix and the product $G^{T}G$ will be a $(m+1)\times(m+1)$ Diagonal matrix. Afterwards a can be calculated with the knows formula $G^{T}Ga = G^{T}y$.

6.3.3 Calculation of the first terms for orthogonal polynomials:

From Equation 33 one knows that $\mathrm{N}^{(i)}$ and $t^{(i)}$ are the following:

$$\mathrm{N}^{(i)} = \underbrace{\underbrace{(\mathrm{N}-1)}_{\mathrm{N}^0}(\mathrm{N}-2)}_{\mathrm{N}^1}(\mathrm{N}-2)\cdots(\mathrm{N}-i+1) \tag{34}$$

$$t^{(i)} = \underbrace{\underbrace{(t-0)}_{t^1}(t-1)}_{t^1}(t-2)\cdots(t-i+1) \tag{35}$$

$$p_{0,4}(t) = \sum_{i=0}^{0} (-1)^0 \binom{0}{0} \binom{0+0}{0} \frac{1 = t^0}{1 = 4^0} = \underline{\underline{1}}$$

$$p_{1,4}(t) = \sum_{i=0}^{1}(-1)^i \binom{1}{i}\binom{1+i}{i}\frac{t^i}{N^i}$$

$$= (-1)^0 \underbrace{\binom{1}{0}}_{=1}\underbrace{\binom{1+0}{0}}_{=1}\frac{1 = t^{(0)}}{1 = 4^0}$$

$$+ (-1)^1 \underbrace{\binom{1}{1}}_{=1}\underbrace{\binom{1+1}{1}}_{=2}\frac{t = t^{(1)}}{4 = 4^1}$$

$$= \underline{\underline{1 - \frac{t}{2}}}$$

$$p_{2,4}(t) = \sum_{i=0}^{2}(-1)^i \binom{2}{i}\binom{2+i}{i}\frac{t^i}{N^i}$$

$$= (-1)^0 \underbrace{\binom{2}{0}}_{=1}\underbrace{\binom{2+0}{0}}_{=1}\frac{1 = t^{(0)}}{1 = 4^0}$$

$$+ (-1)^1 \underbrace{\binom{2}{1}}_{=2}\underbrace{\binom{2+1}{1}}_{=3}\frac{t = t^{(1)}}{4 = 4-0}$$

$$+ (-1)^2 \overbrace{\binom{2}{2}}^{=\frac{2!}{2!(2-2)!}=1}\underbrace{\binom{2+2}{2}}_{=\frac{4!}{2!(4-2)!}=6}\frac{t\cdot(t-1) = t^2 - t}{4\cdot(4-1) = 12}$$

$$= \underline{\underline{1 - \frac{3}{2}\cdot t + \frac{1}{2}\cdot t^2 - \frac{1}{2}\cdot t}}$$

6.3.4 Exercise one, least square parabola

Compute a linear least-squares approximating parabola for the "second window" of five consecutive points (starting with x = 2) in the data.

$$\{\{x, y\}\} = \{\{1, 1.04\}, \{2, 1.37\}, \{3, 1.70\}, \{4, 2.00\}, \{5, 2.26\}$$
$$\{6, 2.42\}, \{7, 2.70\}, \{8, 2.78\}, \{9, 3.00\}, \{10, 3.14\}\}$$

Lets also say the following:

- m = deg = 2 (polys) $\{1; x; x^2\}$
- N=4 (number of arguments actually 5 points)

Note: Normally: m≪N (The degree is smaller than the number of points)

Write down the design matrix and the system of normal equations.

$$G = \left\{ \begin{array}{c|ccc} & g_0 & g_1 & g_2 \\ \hline x_0 & g_0(x_0) & g_1(x_0) & g_2(x_0) \\ x_1 & g_0(x_1) & g_1(x_1) & g_2(x_1) \\ x_2 & \vdots & \vdots & \vdots \\ x_3 & & & \\ x_4 & & & \end{array} \right\} = \left\{ \begin{array}{c|ccc} & 1 & x & x^2 \\ \hline 2 & 1 & 2 & 4 \\ 3 & 1 & 3 & 9 \\ 4 & 1 & 4 & 16 \\ 5 & 1 & 5 & 25 \\ 6 & 1 & 6 & 36 \end{array} \right\}$$

$$\overbrace{\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{pmatrix}}^{\text{Design matrix G}} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1.37 \\ 1.70 \\ 2.00 \\ 2.26 \\ 2.42 \end{pmatrix} \tag{36}$$

With Equation 30 we get the system of normal equations as one can see in Equation 37:

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 4 & 9 & 16 & 25 & 36 \end{bmatrix}}^{\mathbf{G}^T} \cdot \overbrace{\begin{bmatrix} 1.37 \\ 1.7 \\ 2 \\ 2.26 \\ 2.42 \end{bmatrix}}^{y} = \begin{bmatrix} 9.75 \\ 41.66 \\ 196.4 \end{bmatrix}$$

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 4 & 9 & 16 & 25 & 36 \end{bmatrix}}^{G^T G} \cdot \overbrace{\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}}^{G} = \begin{bmatrix} 5 & 20 & 90 \\ 20 & 90 & 440 \\ 90 & 440 & 2274 \end{bmatrix}$$

$$\overbrace{\begin{pmatrix} 5 & 20 & 90 \\ 20 & 90 & 440 \\ 90 & 440 & 2274 \end{pmatrix}}^{G^T G} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 9.75 \\ 41.66 \\ 196.4 \end{pmatrix} \tag{37}$$

Solve the linear system

$$a_0 = 0.506; a_1 = 0.483143; a_2 = -0.0271429$$
$$y = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2$$

When one wants to increase the stability of the matrix one can make a statistical normalization.

Compute the output y and the derivative (!) of the approximation at the central coordinate (x = 4)

$$y(4) = \underline{\underline{2.00429}}; \quad y'(4) = a_1 + 2a_2 \cdot 4 = \underline{\underline{0.266}}$$

6.3.5 Excercise three, Savitzky Golay filter

Apply the filter formulas developed in the exercise before for the data to compute approximately y for $x = 3,\ldots,8$

x_k	1	2	3	4	5	6	7	8	9	10
y_k	1.04	1.37	1.70	2.00	2.26	2.42	2.70	2.78	3.00	3.14

$$
\begin{aligned}
k=2 &\Rightarrow x=3: a_0 = y_2 - \frac{3}{35}\Delta^4 y_0 = 1.7c - \frac{3}{35}0.02 = 1.6983\\
k=3 &\Rightarrow x=4: a_0 = y_3 - \frac{3}{35}\Delta^4 y_1 = 2 - \frac{3}{35}(-0.05) = 2.0043\\
k=4 &\Rightarrow x=5: a_0 = y_4 - \frac{3}{35}\Delta^4 y_2 = 2.26 - \frac{3}{35}(0.28) = 2.236\\
k=5 &\Rightarrow x=6: a_0 = y_5 - \frac{3}{35}\Delta^4 y_3 = 2.42 - \frac{3}{35}(-0.54) = 2.4663\\
k=6 &\Rightarrow x=7: a_0 = y_6 - \frac{3}{35}\Delta^4 y_4 = 2.7 - \frac{3}{35}(0.66) = 2.6434\\
k=7 &\Rightarrow x=8: a_0 = y_7 - \frac{3}{35}\Delta^4 y_5 = 2.78 - \frac{3}{35}(-0.56) = 2.828
\end{aligned}
$$

6.3.6 Exercise four, orthogonal polynomials

Solve subsubsection 6.3.4 again by using the orthogonal polynomials $\{p_{k,\mathrm{N}}(t)\}_{k=0,\ldots,2}$
From Equation 33 and subsubsection 6.3.3 one knows the three basis functions:

$$
\mathrm{P}_{0,4}(t) = 1 = g_0; \quad \mathrm{P}_{1,4}(t) = 1 - \frac{1}{2}t = g_1(t); \quad \mathrm{P}_{2,4}(t) = 1 - 2\cdot t + \frac{1}{2}t^2 = g_2(t)
$$

Now we fist have to find a transformation. The transformation can be written in the follwoing way:

$$
t = \frac{x-2}{1} = x - 2 \in \{0,\cdots,4\}
$$

$$
\mathrm{G} = \left\{\begin{array}{c|ccc} & g_0 & g_1 & g_2 \\ \hline t_0 & g_0(t_0) & g_1(t_0) & g_2(t_0) \\ t_1 & g_0(t_1) & g_1(t_1) & g_2(t_1) \\ t_2 & \vdots & \vdots & \vdots \\ t_3 & & & \\ t_4 & & & \end{array}\right\} = \left\{\begin{array}{c|ccc} & 1 & 1-\frac{1}{2}t & 1-2\cdot t+\frac{1}{2}t^2 \\ \hline 0 & 1 & 1 & 1 \\ 1 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 2 & 1 & 0 & -1 \\ 3 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 4 & 1 & -1 & 1 \end{array}\right\}
$$

$$
\overbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & 1 \end{pmatrix}}^{\text{Design matrix G}} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1.37 \\ 1.70 \\ 2.00 \\ 2.26 \\ 2.42 \end{pmatrix} \tag{38}
$$

With Equation 30 we get the system of normal equations as one can see in Equation 39:

$$
\overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ 1 & -\frac{1}{2} & -1 & -\frac{1}{2} & 1 \end{bmatrix}}^{\mathbf{G}^{\mathrm{T}}} \cdot \overbrace{\begin{bmatrix} 1.37 \\ 1.7 \\ 2 \\ 2.26 \\ 2.42 \end{bmatrix}}^{y} = \begin{bmatrix} 9.75 \\ -1.33 \\ -0.19 \end{bmatrix}
$$

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ 1 & -\frac{1}{2} & -1 & -\frac{1}{2} & 1 \end{bmatrix}}^{\mathrm{G^T G}} \cdot \overbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & 1 \end{bmatrix}}^{\mathrm{G}} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix}$$

$$\overbrace{\begin{pmatrix} 5 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{7}{2} \end{pmatrix}}^{\mathrm{G^T G}} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \overbrace{\begin{pmatrix} 9.75 \\ -1.33 \\ -0.19 \end{pmatrix}}^{y} \tag{39}$$

When we now calculate $(\mathrm{G^T G})^{-1} \cdot y$ one gets $\begin{pmatrix} 1.95 \\ -0.532 \\ -\frac{0.38}{7} \end{pmatrix}$ The result is therefore:

$$\begin{aligned} y(x) &= a_0 \cdot 1 + a_1 \cdot p_{1,4}(t) + a_2 \cdot p_{2,4}(x) \\ y(x=4) &= y(t=2) = a_0 + a_1 \cdot 0 + a_2(-1) = \underline{\underline{2,00429}} \\ y'(x=4) &= \frac{1}{1} y'(x=2) = a_1 p'_{1,4}(2) + a_2 p'_{2,4}(2). \\ &= a_1 \left(-\frac{1}{2}\right) + (-2+2) = \underline{\underline{0.266}} \end{aligned}$$

Which is the same as insection 6.3.4.

6.3.7 Exercise five, singular value decomposition

Examine and compute a least-squares approximative quadratic parabola for the data

x	-2	-1	0	1	2
y	0	1	2	3	1

with respect to the basis functions $\left\{1, -\frac{x}{2}, \frac{x^2}{2} - 1\right\}$ in the following sense:

Compute the design matrix G and the normal matrix. Hint: The normal matrix here is diagonal!

$$\mathrm{G} = \left\{ \begin{array}{c|ccc} & g_0 & g_1 & g_2 \\ \hline x_0 & g_0(x_0) & g_1(x_0) & g_2(x_0) \\ x_1 & g_0(x_1) & g_1(x_1) & g_2(x_1) \\ x_2 & \vdots & \vdots & \vdots \\ x_3 & & & \\ x_4 & & & \end{array} \right\} = \left\{ \begin{array}{c|ccc} & 1 & -\frac{x}{2} & \frac{x^2}{2} - 1 \\ \hline -2 & 1 & 1 & 1 \\ -1 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -1 \\ 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 2 & 1 & -1 & 1 \end{array} \right\}$$

$$\overbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & 1 \end{pmatrix}}^{\text{Design matrix G}} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 1 \end{pmatrix} \tag{40}$$

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ 1 & -\frac{1}{2} & -1 & -\frac{1}{2} & 1 \end{bmatrix}}^{\mathbf{G}^{\mathrm{T}}} \cdot \overbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}}^{y} = \begin{bmatrix} 7 \\ -2 \\ -3 \end{bmatrix}$$

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ 1 & -\frac{1}{2} & -1 & -\frac{1}{2} & 1 \end{bmatrix}}^{\mathrm{G^TG}} \cdot \overbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & -1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & 1 \end{bmatrix}}^{\mathrm{G}} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix}$$

$$\overbrace{\begin{pmatrix} 5 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{7}{2} \end{pmatrix}}^{\mathrm{G^TG}} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \overbrace{\begin{pmatrix} 7 \\ -2 \\ -3 \end{pmatrix}}^{y} \tag{41}$$

Solve the system of normal equations and write down a formula for the approximating parabola.

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{5}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{7}{2} & 0 & 0 & 1 \end{array}\right]$$

divide first row by factor 5.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & \frac{5}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{7}{2} & 0 & 0 & 1 \end{array}\right]$$

Also divide other rows by its factor.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{2}{7} \end{array}\right]$$

$$\overbrace{\begin{pmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \\ 0 & 0 & \frac{2}{7} \end{pmatrix}}^{(\mathrm{G^TG})^{-1}} \cdot \overbrace{\begin{pmatrix} 7 \\ -2 \\ -3 \end{pmatrix}}^{y} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

$$a_0 = \frac{7}{5}, a_1 = -\frac{4}{5}, a_2 = -\frac{6}{7}$$

$$y = a_0 \cdot 1 + a_1 \cdot -\frac{x}{2} + a_2 \frac{x^2}{2} - 1$$

$$y = \frac{7}{5} \cdot 1 + -\frac{4}{5} \cdot -\frac{x}{2} + -\frac{6}{7} (\frac{x^2}{2} - 1)$$

$$y = \frac{7}{5} + \frac{2}{5} \cdot x + \frac{6}{7} - \frac{3}{7} \cdot x^2$$

$$y = \frac{79}{35} + \frac{2}{5} \cdot x - \frac{3}{7} \cdot x^2$$

What are the dimensions of the unitary matrices U, V **, as well as the diagonal matrix** D**, in the singular value decomposition** $\mathrm{G} = \mathrm{UDV}^{tr}$

- U=5x5
- D=5x3
- V=3x3

What are the entries (singular values) in the matrix D from above The singular values are the square-root of the non zero eigenvalues of $\mathbf{G}^{\mathrm{T}} \cdot \mathbf{G}$ and therefore $\{\sqrt{5}; \sqrt{5/2}; \sqrt{7/2}\}$

Give three orthogonal basis polynomials (with respect to the data given) as formulas in the variable x. $\left\{1; -\frac{x}{2}; \frac{x^2}{2} - 1\right\}$ is orthogonal because $\mathbf{G}^{\mathrm{T}} \cdot \mathbf{G}$ is diagonal.

6.4 Chebyshev polynomials

6.4.1 Idea

Approximate a continuous polynomial by a Chebyshev polynomial.

6.4.2 Definition

Chebyshev Polynomials are defined as $\mathrm{T}_n(x) = \cos(n \arccos(x))$ with $(n = 0, 1, \ldots)$ and $(-1 \leq x \leq 1)$. Due to that, most data points are at the edge. The first polynomials can be found in Equation 42.

$$
\begin{aligned}
\mathrm{T}_0 &= 1 & x^0 &= 1 = \mathrm{T}_0 \\
\mathrm{T}_1 &= x & x^1 &= x = \mathrm{T}_1 \\
\mathrm{T}_2 &= 2x^2 - 1 & x^2 &= \frac{1}{2}\mathrm{T}_2 + \frac{1}{2}\mathrm{T}_0 \\
\mathrm{T}_3 &= 4x^3 - 3x & x^3 &= \frac{1}{4}\mathrm{T}_3 + \frac{3}{4}\mathrm{T}_1 \\
\mathrm{T}_4 &= 8x^4 - 8x^2 + 1 & x^4 &= \frac{1}{8}\mathrm{T}_4 + \frac{1}{2}\mathrm{T}_2 + \frac{3}{8}\mathrm{T}_0 \\
\mathrm{T}_5 &= 16x^5 - 20x^3 + 5x & x^5 &= \frac{1}{16}\mathrm{T}_5 + \frac{5}{16}\mathrm{T}_3 + \frac{5}{8}\mathrm{T}_1
\end{aligned}
\tag{42}
$$

Further polynomials can be calculated with the recursion formula $\mathrm{T}_{n+1}(x) = 2x\mathrm{T}_n(x) - \mathrm{T}_{n-1}(n \geq 2)$ with the initial conditions $\mathrm{T}_1(x) = x, \mathrm{T}_0(x) = 1$.

6.4.3 Properties

- The maximal amplitude of a Chebyshev-Polynomials is $\frac{1}{2^n}$ and for a normalized one $\frac{1}{2^n}\mathrm{T}_{n+1}(x)$
- Amplitude: $\mathrm{T}_n(x) \in [-1, +1]$
- Zero points: $\mathrm{T}_n(x) = 0 \Leftrightarrow x = \cos\left(\frac{2i+1}{2n}\pi\right) \quad i = 0, 1, \ldots, n-1$ (also called chebyshev knots)
- $\mathrm{T}_n(x) = \pm 1 \Leftrightarrow x = \cos\left(\frac{i\pi}{n}\right) (i = 0, 1, \ldots, n)$

6.4.4 Usage

For Chebychev we use not g but T_x as function.

$$G = \overbrace{\left\{\begin{array}{c|ccccc} & T_0(x) & T_1(x) & T_2(x) & \cdots & T_m(x) \\ \hline x_0 & 1 & x_0 & 2\cdot x_0^2 - 1 & & \\ x_1 & 1 & x_1 & 2\cdot x_1^2-1 & & \\ x_2 & 1 & x_2 & 2\cdot x_2^2 - 1 & & \\ x_3 & \vdots & \vdots & \vdots & & \\ x_N & 1 & x_N & 2\cdot x_N^2-1 & & \end{array}\right\}}^{DesignMatrix} = \begin{bmatrix} T_0(x_0)=1 & T_1(x_0)=x_0 & \dots & T_m(x_0) \\ T_0(x_1)=1 & T_1(x_1)=x_1 & \dots & T_m(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_N)=1 & T_1(x_N)=x_N & \dots & T_m(x_N)\end{bmatrix}_{N\times m}$$

The matrix G^TG can be callculated accroding to Equation 43.

$$\begin{aligned} \langle T_j, T_k\rangle &:= \sum_{i=0}^{N} T_j(x_i)\,T_k(x_i) = \begin{cases} 0 & j\neq k \\ (N+1)/2 & j=k\neq 0 \\ N+1 & j=k=0\end{cases} \quad (j,k=0,\dots,N) \\ x_i &= \cos\left(\frac{2i+1}{2(N+1)}\pi\right) \quad (i=0,1,\dots,N) \end{aligned} \tag{43}$$

And results then in the follwing matrix

$$G^TG \underbrace{=}_{\text{When Chebyshev}} \begin{bmatrix} N+1 & 0 & \cdots & 0 \\ 0 & \frac{N+1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{N+1}{2}\end{bmatrix}_{m\times m}$$

$$\left(G^TG\right)^{-1}G^T \underbrace{=}_{\text{When Chebyshev}} \begin{bmatrix} \frac{1}{N+1}T_0(x_0) = \frac{1}{N+1} & \frac{1}{N+1}T_0(x_1)=\frac{1}{N+1} & \cdots & \frac{1}{N+1}T_0(x_N) = \frac{1}{N+1} \\ \frac{2}{N+1}T_1(x_0) = \frac{2}{N+1}x_0 & \frac{2}{N+1}T_1(x_1) = \frac{2}{N+1}x_1 & \cdots & \frac{2}{N+1}T_1(x_N)=\frac{2}{N+1}x_N \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N+1}T_m(x_0) & \frac{2}{N+1}T_m(x_1) & \cdots & \frac{2}{N+1}T_m(x_N)\end{bmatrix}_{m\times N}$$

Recipe Goal: $y(t) = p_N(t)$ (define polynomial in $[a,b]$) with Chebyshev-polynomials of degree m.

1. transform $y(t)$ on the standard interval $[-1,1]$ (affine Transformation) $t = a + \frac{b-a}{2}(x+1)$.
2. describe $y(x)$ with T_n terms
3. truncate $y(x)$ on degree m : $y_m(x)$
4. back transformation: $x = 2\frac{t-a}{b-a} - 1$
5. insert T_n in y_m see Equation 42
6. error estimation for truncate Method: $\max_t |y(t) - y_m(t)|$ (removed part)

Example Approximation of $y(t) = t^3$ with degree $m = 2$ for the interval $(a,b) = (0,1)$

1. Transformation with $t = \frac{x+1}{2}$

$$y(x) = \left(\frac{x+1}{2}\right)^3 = \frac{1}{8}\left(x^3+3x^2+3x+1\right)$$

2. Expand with T_n see also Equation 42:

$$\begin{aligned} y(x) &= \frac{1}{8}\left(\frac{T_3(x)+3T_1(x)}{4} + 3\frac{T_2(x)+T_0}{2} + 3T_1(x) + T_0(x\right. \\ &= \frac{1}{32}T_3(x) + \frac{3}{16}T_2(x) + \frac{15}{32}T_1(x) + \frac{5}{16}T_0(x) \end{aligned}$$

3. shorten to degree $m = 2$: $y(x) \approx \frac{3}{16}T_2(x) + \frac{15}{32}T_1(x) + \frac{5}{16}T_0(x)$

4. back transformation with $x = 2t-1$: $y(t) \approx \frac{3}{16}T_2(2t-1) + \frac{15}{32}T_1(2t-1) + \frac{5}{16}T_0(2t-1)$

5. $T_n(2t-1)$ substitution: $y(t) \approx \frac{3}{16}\left(2(2t-1)^2-1\right) + \frac{15}{32}(2t-1) + \frac{5}{16}$

6. error estimation: $\max_t \left|\frac{1}{32}T_3(2t-1)\right|$

6.5 Continuous Chebyshev approximation

A weight function $w(x)$ is often called the function inside the integral, in the case of the formula below $w(x) = \frac{1}{\sqrt{1-x^2}}$.

$$\langle T_j, T_k\rangle_{\text{cont}} := \int_{-1}^{1} T_j(x)T_k(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & j \neq k \\ \pi/2 & j = k \neq 0 \\ \pi & j = k = 0 \end{cases} \tag{44}$$

Equation 44 is true because the polynomial is orthogonal!

Example:Chebyshev continuous least-squares parabola on the interval $[0,1]$ **for** $y(t) = t^3$ **First the interval** $[0,1]$ **is transformed to** $[-1,1]$ **by** $x = 2t-1$ This is shown with **??**

$$-1 + 2\frac{x-a}{b-a} = -1 + 2\frac{x-0}{1-0} = 1 - 2x$$

In our case our new x is called t. Therefore $x = 2t-1 \Rightarrow t = \frac{x+1}{2} \Rightarrow y(t) = \frac{(x+1)^3}{8}$ Now we can use Equation 45 and get the following results:

$$a_0 = \frac{1}{\pi}\int_{-1}^{1} y(x)\frac{dx}{\sqrt{1-x^2}} = \frac{5}{16}$$
$$a_1 = \frac{2}{\pi}\int_{-1}^{1} y(x)T_1(x)\frac{dx}{\sqrt{1-x^2}} = \frac{15}{32}$$
$$a_2 = \frac{2}{\pi}\int_{-1}^{1} y(x)T_2(x)\frac{dx}{\sqrt{1-x^2}} = \frac{3}{16}$$

$$p(x) = \sum_{j=0}^{m} a_j T_j(x) \quad \text{wobei} \quad a_j = \begin{cases} \frac{1}{\pi}\int_{-1}^{1} \frac{y(x)}{\sqrt{1-x^2}}dx & j = 0 \\ \frac{2}{\pi}\int_{-1}^{1} \frac{y(x)T_j(x)}{\sqrt{1-x^2}}dx & j > 0 \end{cases} \tag{45}$$

6.6 Continuous Least-Square Legendre approximation

Legendre Polynomials: The Legendre polynomials are defined by the Rodriguez formula which can be seen in Equation 46

$$P_n(x) = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}\left(x^2-1\right)^n \tag{46}$$

Where the first polynomials can be seen in Equation 47

$$\begin{gathered} P_n(x) = \frac{1}{2^n n!}\cdot\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^2 \quad P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}\left(3x^2-1\right) \\ P_3(x) = \frac{1}{2}\left(5x^3-3x\right) \quad P_4(x) = \frac{1}{8}\left(35x^4-30x^2+3\right) \quad P_5(x) = \frac{1}{8}\left(63x^5-70x^3+15x\right) \end{gathered} \tag{47}$$

Continuous Legendre least-squares approximation
If $y(x)$ is function on $[-1,1]$ which is absolutely square-integrable with respect to the weight function $w(x) = 1 \quad (x \in [-1,1])$ in the sense that $\int_{-1}^{1} |y(x)|^2 dx < \infty$, then the continuous square-sum of residuals in Equation 48

$$\mathrm{S} := \int_{-1}^{1} \left(y(x) - \sum_{j=0}^{m} a_j \mathrm{P}_j(x) \right)^2 dx \tag{48}$$

Is minimal when the coefficients a have the value given in Equation 49. This equation also describes the resulting polynomial.

$$p(x) = \sum_{j=0}^{m} a_j \mathrm{P}_j(x) \quad \text{whereas} \quad a_j = \frac{2j+1}{2} \cdot \int_{-1}^{1} y(x) \mathrm{P}_j(x) \cdot \mathrm{d}x \quad (j = 0, 1, \ldots, m) \tag{49}$$

where m is the degree.

6.6.1 Legendre continuous least square parabola

Legendre continuous least-squares parabola on the interval $t \in [0,1]$ for $y(t) = t^3$.

First one has to do a transformation to the interval $x \in [-1,1]$. This results in the following: $x = 2t - 1 \Rightarrow t = \frac{x+1}{2}$, therefore $y(x) = \frac{(x+1)^3}{8}$. With Equation 49 one gets then the following:

$$\begin{aligned}
a_0 &= \frac{1}{2} \int_{-1}^{1} \underbrace{y(x)}_{\frac{(x+1)^3}{8}} \underbrace{\mathrm{P}_0(x)}_{1} dx = \frac{1}{2} \int_{-1}^{1} y(x) dx = \frac{1}{4} \\
a_1 &= \frac{3}{2} \int_{-1}^{1} y(x) \mathrm{P}_1(x) dx = \frac{3}{2} \int_{-1}^{1} y(x) x dx = \frac{9}{20} \\
a_2 &= \frac{5}{2} \int_{-1}^{1} y(x) \mathrm{P}_2(x) dx = \frac{5}{2} \int_{-1}^{1} y(x) \left(\frac{3x^2 - 1}{2} \right) dx = \frac{1}{4}
\end{aligned}$$

6.7 Mulit-variate least-square

Note: the degree of a multi-variate polynomial can be identified by adding up the degrees of the variables in each of the terms. It does not matter that there are different variables. The largest number is the degree.

- Input bi-variate (x,y) 2D : $\vec{x}$
- Output scalar Z: 1D
- Degree 3.

$$\left\{1, x, y, x^2, 2xy, y^2, x^3, 3x^2y, 3xy^2, y^3\right\}$$

- Degree 4.

$$\left\{1, x, y, x^2, 2xy, y^2, x^3, 3x^2y, 3xy^2, y^3, x^4, 4x^3y, 6x^2y^2, 4xy^3, y^4\right\}$$

Residuals: $r_i = z_i - f(\vec{x}_i) \Rightarrow \sum r_i^2 \rightarrow min!$

$$G = \overbrace{\left\{\begin{array}{c|ccc} & g_0 & g_1 & g_2 \\ \hline \vec{x}_0 & g_0(x_0) & g_1(x_0) & g_2(x_0) \\ \vec{x}_1 & g_0(x_1) & g_1(x_1) & g_2(x_1) \\ \vec{x}_2 & \vdots & \vdots & \vdots \\ \vdots & & & \\ \vec{x}_N & & & \end{array}\right\}}^{\text{Design matrix}} = \left\{\begin{array}{c|ccccc} & 1 & y & y^2 & \cdots & xy^2 \\ \hline \vec{x}_0 & 1 & y_0 & y_0^2 & & \vdots \\ \vec{x}_1 & 1 & y_1 & y_1^2 & & \\ \vec{x}_2 & \vdots & \vdots & \vdots & & \\ \vdots & & & & & \\ \vec{x}_N & 1 & y_N & y_N^2 & & \end{array}\right\}$$

Basis Functions d=dim=2

$$\sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \cdots \sum_{j_d=0}^{m_d} a_{j_1,j_2,\ldots,j_d} g_{j_1}\left(x^{(1)}\right) g_{j_2}\left(x^{(2)}\right) \cdots g_{j_d}\left(x^{(d)}\right)$$

Product

$$\underbrace{\{1,x\}}_{\text{'2'}} \, x \, \underbrace{\{1,y,y^2\}}_{\text{'3'}}$$

$$\{1,y,y^2,x,xy,xy^2\} = \text{'6'}$$

Statistical norm

$$\left\{\left(\frac{x^{(1)}-\mu_1}{\sigma_1}\right)^j \left(\frac{x^{(2)}-\mu_2}{\sigma_2}\right)^k \mid j,k \in \mathrm{N}_0\right\}$$

6.7.1 Example one

Normally, a set of four 3d-points (x,y,z) is not contained in one single plane. But generally there is a plane coming close to the 3d-points in the sense of least-squares approximation.
The (x,y,z)-data in this problem is: A = (1,0,0), B = (0,1,0), C = (0,2,-1), D = (1,3,1)

(a) Give a reasonable set of basis functions. Hint: A plane has total degree 1 (complete basis)

As a basis function, one can use:

$$\{1,x,y\}$$

(b) Write down the design matrix according to a) and the normal equations

$$G = \overbrace{\left\{\begin{array}{c|ccc} & g_0 & g_1 & g_2 \\ \hline \vec{x}_0 & g_0(\vec{x}_0) & g_1(\vec{x}_0) & g_2(\vec{x}_0) \\ \vec{x}_1 & g_0(\vec{x}_1) & g_1(\vec{x}_1) & g_2(\vec{x}_1) \\ \vec{x}_2 & \vdots & \vdots & \vdots \\ \vdots & & & \\ \vec{x}_N & & & \end{array}\right\}}^{\text{Design matrix}} = \left\{\begin{array}{c|ccc} & 1 & x & y \\ \hline (1,0) & 1 & 1 & 0 \\ (0,1) & 1 & 0 & 1 \\ (0,2) & 1 & 0 & 2 \\ (1,3) & 1 & 1 & 3 \end{array}\right\}$$

(c) Solve the system of normal equations and give a functional formula for the approximating plane.

According to Equation 30 to following is true: $\underbrace{\vec{a} = \left(G^T \cdot G\right)^{-1} G^T \cdot y}_{\text{normal equations}}$

$$\vec{a} = \begin{bmatrix} -\frac{4}{5} \\ 1 \\ \frac{1}{5} \end{bmatrix}$$

$$f(x,y) = -\frac{4}{5} + x + \frac{1}{5}y$$

6.7.2 Example three

Express x^3, x^4 as linear combinations of the Chebyshev polynomials $T_0(x), T_1(x), T_2(x), T_3(x), T_4(x)$.

From Equation 42 one knows that $T_3(x) = 4x^3 - 3x \Rightarrow \frac{1}{4}T_3(x) = x^3 - \frac{3}{4}x \Rightarrow \underline{\underline{\frac{1}{4}T_3(x) + \frac{3}{4}T_1(x)}} = x^3$

From Equation 42 one knows that $T_4(x) = 8x^4 - 8x^2 + 1 \Rightarrow \frac{1}{8}T_4(x) = x^4 - x^2 + \frac{1}{8} \Rightarrow \frac{1}{8}T_4(x) + \frac{1}{2}T_2(x) = x^4 - \frac{3}{8} \Rightarrow$ $\underline{\underline{\frac{1}{8}T_4(x) + \frac{1}{2}T_2(x) + \frac{3}{8}T_0}} = x^4$

6.7.3 Example six

Express x^3 as linear combinations of the Legendre polynomials $P_0(x), P_1(x), P_2(x), P_3(x)$.
From Equation 47 one knows that $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) \Rightarrow \frac{2}{5}P_3(x) = x^3 - \frac{3}{5}x \Rightarrow \underline{\underline{\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)}} = x^3$

6.7.4 Example seven

Compute continuously approximating least-squares lines for the model function $y(t) = t^2$ $(0 \geq t \geq 1)$ by

(a) Chebyshev approximation with the weight function $w(x) = 1/\sqrt{1-x^2} \quad (-1 < x < 1)$

Firstly one has to bring it into the correct range. One can do that with **??**
Map the interval [a,b] onto the interval [c,d]

$$f(t) = c + \left(\frac{d-c}{b-a}\right)(t-a) \tag{50}$$

$$\underbrace{f(t)}_{x} = -1 + \left(\frac{1+1}{1-0}\right)(t-0) = -1 + 2t \Rightarrow x = -1 + 2t \Rightarrow t = \frac{1}{2}(x+1)$$

$$y = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}$$

$$y = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}$$

$$\frac{1}{8}T_2 = \frac{1}{4}x^2 - \frac{1}{8} \Rightarrow \frac{1}{8}T_2 + \frac{1}{2}T_1 + \frac{3}{8}T_0 \Rightarrow \underline{line = \frac{3}{8} + \frac{1}{2}x} = \underline{\underline{\frac{3}{8} + \frac{1}{2}(2t-1)}}$$

(b) Legendre approximation with the weight function $w(x) = 1$

(c) Estimate the maximum approximating errors in ab) by the coefficients of the Chebyshev polynomial $T_2(x)$ and Legendre polynomial $P_2(x)$, respectively

7 Differentials, Taylor formulas and Jacobian

7.1 Differential

7.1.1 Definition

The purpose of differential is to measure error propagation. (How much is y(dependent variable) wrong when x(independent variable) is wrong by a certain amount). The differential df is the linear amount of change between a variable and a function, as it can be seen in Equation 51. Whereby Δf is the hole amount of change, not only the linear one (The difference between two points). For small dx on can say $\Delta f \approx df$ and $\frac{\Delta f}{f} \approx \frac{df(x_0)}{f(x_0)}$.

$$\Delta f = f(x_0 + h) - f(x_0) \approx df = f'(x_0)\,dx = f'(x_0)\,h = f'(x_0)\,\Delta x \tag{51}$$

7.2 Taylor

As one has seen before, $\Delta f \approx df$ for small dx and one has used only the linear part. To further improve the approximation one could not only use the linear part (first derivative) but also the squared (second derivative) and so on. Therefore, a function at a certain point can be approximated by it's derivates at this point, which is called Taylor series approximation, which can be seen in Equation 52 for the one dimensional case and in Equation 53 for the multidimensional case.

$$\Delta f = \frac{1}{1!}df(x_0) + \frac{1}{2!}d^2 f(x_0) + \ldots + \frac{1}{n!}d^n f(x_0) + \mathrm{R}_n(x_0, h) \tag{52}$$

The vector field of of the partial derivative is called gradient of f and is denoted with grad(f) or $\vec{\nabla} f(\vec{x})$. Therefore $\Delta f \approx \vec{\nabla} f(\vec{x}) \cdot \vec{h}$. Equation 53 shows the taylor series approximation for the multi-indices $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

$$f\left(\vec{x}_0 + \vec{h}\right) = f(\vec{x}_0) + \frac{1}{1!}\vec{\nabla} f(\vec{x}_0) \cdot \vec{h} + \sum_{|\alpha|=2}^{\mathrm{N}} \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f(\vec{x}_0)}{\partial x^{\alpha}}\vec{h}^{\alpha} + \sum_{|\alpha|=\mathrm{N}+1} \mathrm{R}_{\alpha}\left(\vec{x}_0, \vec{h}\right)\vec{h}^{\alpha} \tag{53}$$

The remainder terms $\mathrm{R}_{\alpha}\left(\vec{x}_0, \vec{h}\right)$ are absolutely bounded by $\max_{\vec{x}\in \mathrm{S}}\left|\frac{1}{\alpha!}\frac{\partial^{\alpha} f(\vec{x})}{\partial x^{\alpha}}\right|$ with $|\alpha| = \mathrm{N}+1$ and $\mathrm{S} = \vec{x}_0 + ([-h_1, h_1] \times [-h_2, h_2] \times \cdots \times [-h_n, h_n])$ is a n-dimensional 'rectangle' with center $\vec{x}_0$. (This idea can be used afterwards by the Jacobian matrix and the determinant)
The formulas for up to order four can be found in Equation 54

$$\begin{aligned}
&f(x,y) = f(0+x, 0+y) \approx \\
&f(0,0) + \frac{\partial f}{\partial x}\cdot x + \frac{\partial f}{\partial y}\cdot y + \frac{\partial^2 f}{\partial x^2}\cdot\frac{1}{2!}x^2 + \frac{\partial^2 f}{\partial x \partial y}\cdot\frac{1}{1!1!}\cdot xy + \frac{\partial^2 f}{\partial y^2}\cdot\frac{1}{2!}\cdot y^2 + \\
&+ \frac{\partial^3 f}{\partial x^3}\cdot\frac{1}{3!}\cdot x^3 + \frac{\partial^3 f}{\partial x^2 \partial y}\cdot\frac{1}{2!1!}\cdot x^2 y + \frac{\partial^3 f}{\partial x \partial y^2}\cdot\frac{1}{1!2!}\cdot xy^2 + \frac{\partial^3 f}{\partial y^3}\cdot\frac{1}{3!}\cdot y^3 + \\
&+ \frac{\partial^4 f}{\partial x^4}\cdot\frac{1}{4!}x^4 + \frac{\partial^4 f}{\partial x^3 \partial y}\cdot\frac{1}{3!1!}\cdot x^3 y + \frac{\partial^4 f}{\partial x^2 \partial y^2}\cdot\frac{1}{2!2!}\cdot x^2 y^2 + \frac{\partial^4 f}{\partial x \partial y^3}\cdot\frac{1}{1!3!}\cdot xy^3 + \\
&+ \frac{\partial^4 f}{\partial y^4}\cdot\frac{1}{4!}\cdot y^4
\end{aligned} \tag{54}$$

7.2.1 Example

The bivariate symmetric (!) function $f(x,y) = e^{-\frac{x^2+y^2}{2}}$ has to be approximated by a bivariate Taylor polynomial of order 4 around (0,0) by

1. evaluating and using Equation 53 for the partial derivatives.

$$\begin{aligned}&1+0x+0y-\frac{1}{2}x^2+0xy-\frac{1}{2}y^2+0x^3+0x^2y+0xy^2\\&+0y^3+\frac{3}{4!=24}x^4+0x^3y+\frac{1}{2\cdot 2}x^2y^2+0xy^3+\frac{3}{24}y^4\\&=\underline{\underline{1-\frac{1}{2}x^2-\frac{1}{2}y^2+\frac{3}{24}x^4+\frac{1}{4}x^2y^2+\frac{3}{24}y^4}}\end{aligned}$$

2. multiplying the univariate Taylor series for the exponential function,

$$\begin{aligned}&f=e^{-\frac{x^2}{2}}\cdot e^{-\frac{y^2}{2}}\approx\left(1-\frac{x^2}{2}+\frac{x^4}{4\cdot 2}+\ldots\right)\cdot\left(1-\frac{y^2}{2}+\frac{y^4}{4\cdot 2}+\ldots\right)\ldots\\&\left(\text{by substituting } u=-\frac{x^2}{2} \text{ and } u=-\frac{y^2}{2}\text{ , resp.}\right)\\&\ldots=\underline{\underline{1-\frac{x^2}{2}-\frac{y^2}{2}+\frac{x^4}{8}+\frac{1}{4}x^2y^2+\frac{y^4}{8}+\cdots)}}\end{aligned}$$

3. substituting into the univariate Taylor series for the exponential function

$$\begin{aligned}&\text{Subst: } u=-\frac{x^2+y^2}{2}\Rightarrow f\approx 1-\frac{x^2+y^2}{2}+\frac{1}{2}\left(\frac{x^2+y^2}{2}\right)^2\\&=\underline{\underline{1-\frac{x^2}{2}-\frac{y^2}{2}+\frac{1}{8}x^4+\frac{1}{4}x^2y^2+\frac{1}{8}y^4}}\end{aligned}$$

4. What value do you expect for the error limit: $\lim_{\|h\|\to 0}\frac{\text{"Approximation error}}{\|h\|^4}$

$$\begin{aligned}&\lim_{\|h\|\to 0}\frac{\text{"Error"}}{\|h\|^4}=0\\&\left(\|h\|=\sqrt{h_1^2+h_2^2}=\sqrt{x^2+y^2}\right)\\&\text{In c}):-\tfrac{x^2+y^2}{2}=u=-\tfrac{\|h\|^2}{2}(!).\\&e^h=f\approx\underbrace{1-\frac{\|h\|^2}{2}+\frac{\|h\|^4}{8}}_{\text{"4th order}}-\tfrac{\|h\|^6}{48}+-\ldots\\&\Rightarrow\text{ Error }=-\frac{\|h\|^6}{48}+-\cdots\\&\Longrightarrow\frac{\text{Error}}{\|h\|^4}=-\frac{\|h\|^2}{48}+-\overrightarrow{\|h\|\to c0\text{ ob}}\text{)}\end{aligned}$$

7.3 Jacobian matrix and determinant

In the Taylor series approximation for multi-indices, one has seen that $\Delta f \approx \vec{\nabla} f(\vec{x}) \cdot \vec{h}$ for small dx (see also Equation 53). When one writes all those gradients in one matrix, one gets the so-called Jacobian matrix. Which actually tells us the same as mentioned in subsubsection 7.1.1, but just for a multivariable problem. It tells us how $y_1, y_2, \ldots$ (dependent variable) changes when, $x_1, x_2, \ldots$ (independent variable) changes. Furthermore it has the nice property that the determinant of this matrix also describes how the volume changes when one changes a certain variable.Therefore the Matrix can be used to analyse the error propagation or make volume calculation in a different coordinate system.
Below one can find some definitions.

$$\frac{\partial \vec{f}(u,v)}{\partial u} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \end{pmatrix} \text{ and } \frac{\partial f(u,v)}{\partial v} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \end{pmatrix} \tag{55}$$

$$\underbrace{dy_1 dy_2 \cdots dy_n}_{\text{"transformed volume element"}} = \det\left(\mathrm{J}_f(\vec{x})\right) \underbrace{dx_1 dx_2 \cdots dx_n}_{\text{volume element}} \tag{56}$$

$$\underbrace{\Delta y_1 \Delta y_2 \cdots \Delta y_n}_{\text{"transformed volume"}} \approx \det\left(\mathrm{J}_f(\vec{x})\right) \underbrace{\Delta x_1 \Delta x_2 \cdots \Delta x_n}_{\text{"volume"}} \tag{57}$$

In the special case that n = m the Jacobian matrix is a square matrix and thus has a determinant (called Jacobian determinant):

$$\frac{\mathrm{D}\left(f_1, f_2, \ldots, f_n\right)}{\mathrm{D}\left(x_1, x_2, \ldots, x_n\right)} = \det\left(\mathrm{J}_f(\vec{x})\right) \quad \text{or} \quad \det\left(\mathrm{J}_f\right) = \sqrt{\det(\underbrace{\mathrm{J}_f^{\mathrm{T}}(\vec{x}) \mathrm{J}_f(\vec{x})}_{\text{Diagonalmatrix}})}. \tag{58}$$

$$\mathrm{T}^{-1} : \det\left(\mathrm{J}_{\mathrm{T}}\right) = \frac{1}{\det \mathrm{J}_{\mathrm{T}^{-1}}} \tag{59}$$

$$\mathbb{J} = \begin{bmatrix} \frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^{\mathrm{T}} f_1(\mathbf{x}) \\ \vdots \\ \nabla^{\mathrm{T}} f_m(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial f_m(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

7.3.1 Estimating navigation error by inversion of Jacobian determinant

Lets assume one measures an angle (α, γ) and want the position (x, y). Let's call the transformation from (α, γ) to (x, y) T and from (x, y) to (α, γ) T^{-1}

$$\begin{aligned} &\mathrm{B} = (-40, -40) = \left(x_1, y_1\right) \\ &\mathrm{C} = (-40, 2140) = \left(x_2, y_2\right) \\ &\mathrm{A} = (3040, 1050) = \left(x_3, y_3\right) \\ &a = 2180, c = 3267.19 \\ &\mathrm{P} = (x, y) \\ &\Delta\alpha = 0.1^\circ, \Delta\gamma = 0.1^\circ \end{aligned}$$

To solve this problem one can follow the following approach:

1. Description of the problem

2. Transformation (equalities) from cosine-theorems (generalized Pythagoreans)

3. Implicit differentiation
4. Computation of the Jacobian matrix $J_{T^{-1}}(x, y)$
5. Computation of the Jacobian determinant $\left|J_{T^{-1}}(x, y)\right|$
6. Elimination of angles
7. Computing $\left|J_T(\alpha, \gamma)\right|$ expressed in position coordinates x, y

7.3.2 Example three

The formulas $x = r\cos(\varphi), \quad y = r\sin(\varphi) \quad (0 < r, \quad 0 \leq \varphi < 2\pi)$ define the coordinate transform T from polar coordinates (r, ϕ) to Cartesian coordinates (x, y) in the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$.
Compute the Jacobian matrix $J_T(r, \varphi)$ and its Jacobian (determinant) $\det J_T(r, \varphi)$ as well as the Jacobians (determinants) $\det J_T(x, y), \det J_{T^{-1}}(r, \varphi)$ and $\det J_{T^{-1}}(x, y)$

First of all one has to write down the transformations

$$\begin{aligned} x &= r \cdot \cos\phi \\ y &= r \cdot \sin\phi \end{aligned}$$

Once one has done that one can calculate the Jacobian Matrix:

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{bmatrix} \\ &= \underline{\underline{\begin{bmatrix} \cos\phi & -r \cdot \sin\phi \\ \sin\phi & r \cdot \cos\phi \end{bmatrix}}} \end{aligned}$$

and afterwards the determinant

$$\begin{aligned} \det J_T(r, \varphi) &= r \cdot \cos\phi^2 + r \cdot \sin\phi^2 \\ &= r \cdot \left(\cos\phi^2 + \sin\phi^2\right) \\ &= \underline{\underline{r}} \end{aligned}$$

when one wants to express r in x and y one knows that $r = \sqrt{x^2 + y^2}$ therefore

$$\underline{\underline{\det J_T(x, y) = \sqrt{x^2 + y^2}}}$$

and

$$\begin{aligned} \det J_T(r, \varphi) &= \underline{\underline{\frac{1}{r}}} \\ \det J_{T^{-1}}(x, y) &= \underline{\underline{\frac{1}{\sqrt{x^2 + y^2}}}} \end{aligned}$$

7.3.3 Example one

Elliptical coordinates (σ, τ) $(\sigma > 1, -1 < \tau < 1)$ for a = 1 are connected to Cartesian coordinates (x, y) through the transforming formulas:

$$x = \sigma\tau$$
$$y = \sqrt{\sigma^2 - 1}\sqrt{1 - \tau^2}$$

Compute the Jacobian matrix

$$J = J(\sigma, \tau)$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \sigma} & \frac{\partial x}{\partial \tau} \\ \frac{\partial y}{\partial \sigma} & \frac{\partial y}{\partial \tau} \end{bmatrix} = \begin{bmatrix} \tau & \sigma \\ \frac{1}{2}\left(\sigma^2 - 1\right)^{-\frac{1}{2}} 2\sigma\sqrt{1-\tau^2} & -\frac{1}{2}\left(1-\tau^2\right)^{-\frac{1}{2}} 2\tau\sqrt{\sigma^2-1} \end{bmatrix} = \underline{\underline{\begin{bmatrix} \tau & \sigma \\ \frac{\sigma\sqrt{1-\tau^2}}{\sqrt{\sigma^2-1}} & -\frac{\tau\sqrt{\sigma^2-1}}{\sqrt{1-\tau^2}} \end{bmatrix}}}$$

Express the Jacobian determinant

$$det(J) = -\frac{\tau^2\sqrt{\sigma^2-1}}{\sqrt{1-\tau^2}} - \frac{\sigma^2\sqrt{1-\tau^2}}{\sqrt{\sigma^2-1}} = \frac{-\tau^2\left(\sigma^2-1\right) - \sigma^2\left(1-\tau^2\right)}{\sqrt{1-\tau^2}\sqrt{\sigma^2-1}} = \underline{\underline{\frac{\tau^2-\sigma^2}{\sqrt{1-\tau^2}\sqrt{\sigma^2-1}}}}$$

Express the Jacobian determinant in x, y, since the denumerator is exactly y, we know it already. Furthermore the following is true: $\sqrt{\left(1+x^2+y^2\right)^2 - 4x^2} = \sigma^2 - \tau^2$

$$det(J) = \underline{\underline{\frac{-\sqrt{\left(1+x^2+y^2\right)^2 - 4x^2}}{y}}}$$

8 Ordinary differential equations

8.1 Definition

An ODE is a differential equation where it's derivatives belong to only one variable. Furthermore it is possible that an ODE can not be solved explicitly, due to that in this chapter it will be investigated how they can be solved numerically.
A first order differential value problem can be expressed like the one in Equation 60

$$y'(x) = f(x, y(x)) \text{ with initial condition } y(x_0) = y_0 \tag{60}$$

8.2 Explicit methods

One way to solve the problem numerically is by tailor series approximation, as it can be seen in Equation 61. (Remember that the tailor series is evaluated from one single point). The idea is that one creates a tailor series approximation up to order p at the starting point and moves then with a step size h to the new position one got from the tailor series approximation and does there the same until one reaches the destination.

$$y(x+h) = y(x) + \frac{y'(x)}{1!}h + \frac{y''(x)}{2!}h^2 + \frac{y'''(x)}{3!}h^3 + \frac{y^{(4)}(x)}{4!}h^4 + \dots + \frac{y^{(p)}(x)}{p!}h^p + \underbrace{\frac{y^{(p+1)}(\xi)}{(p+1)!}h^{p+1}}_{\text{remaining term}} \tag{61}$$

The calculations up to order three (p=3) are given in Equation 62:

$$\begin{aligned} y(x+h) = & y(x) + \frac{f(x,y(x))}{1!}h + \\ & \frac{1}{2!}\left(\frac{\partial f(x,y(x))}{\partial x}1 + \frac{\partial f(x,y(x))}{\partial y}f(x,y(x))\right)h^2 + \\ & \frac{1}{3!}\left(\frac{\partial^2 f(x,y(x))}{\partial x^2}1 + 2\frac{\partial^2 f(x,y(x))}{\partial x \partial y}f(x,y(x)) + \frac{\partial^2 f(x,y(x))}{\partial y^2}f(x,y(x))^2 + \left(\frac{\partial f(x,y(x))}{\partial y}\right)^2 f(x,y(x)) + \frac{\partial f(x,y(x))}{\partial x}\frac{\partial f(x,y(x))}{\partial y}\right)h^3 + \dots + \\ & \frac{1}{4!}y^{(4)}(x)h^4 + \dots + \frac{1}{p!}y^{(p)}(x)h^p + \underbrace{\frac{1}{(p+1)!}y^{(p+1)}(\xi)h^{p+1}}_{\text{remaining term}} \end{aligned} \tag{62}$$

8.2.1 Euler method

The Euler method is a special case of the explicit methods where $p = 1$ and $h = const$. The Formulas to calculate it can be found in Equation 63.

$$\begin{gathered} y_0 = y(x_0) \\ y(x_0+h) \approx y_1 = y_0 + f(x_0, y_0)\,h \approx y(x_0) + y'(x_0)\,h \\ y(x_1+h) \approx y_2 = y_1 + f(x_1, y_1)\,h \approx y(x_1) + y'(x_1)\,h \\ \vdots \\ y(x_{n-1}+h) \approx y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\,h \approx y(x_{n-1}) + y'(x_{n-1})\,h \end{gathered} \tag{63}$$

8.2.2 Error Calculation

- **Global Error:**

$$\max_{0 \le i \le k} |y_i - y(x_i)| \tag{64}$$

- **Local Error:**

$$y(x_n+h) - y_{n+1} = \underbrace{y(x+h)}_{\text{true output}} - \underbrace{(y(x) + y'(x)h)}_{\text{approx. output}} = h \cdot \tau_h(x_n) \tag{65}$$

- **Local slope Error:**

$$\tau_h(x_n) := \frac{y(x_n+h) - y(x_n)}{h} - \left(\frac{y'(x_n)}{1!} + \frac{y''(x_n)}{2!} h^1 + \cdots \frac{y^{(p)}(x_n)}{p!} h^{p-1} \right) \tag{66}$$

8.2.3 Example

Solve the initial value problem $y' = xy^{1/3}$ with $y(1) = 1$ numerically by the method of Taylor with order p = 4 and fixed step-size h = 0.1 for the x-values 1.1 and 1.2 (two steps). All final (!) results should be rounded to the 10th digit. Furthermore, compute the local error (slope) as well as the global error for the two steps. Note that the exact solution of the equation is given by Equation 67.

$$y = \left(\frac{x^2+2}{3} \right)^{3/2} \tag{67}$$

To find out x at 1.1 one firstly needs to calculate the it's tailor series approximation at the starting point.

$$\begin{aligned}
&x = 1, y = 1 && y_0 = 1 \\
&y_1 = xy^{1/3} = 1 \\
&y_2 = y^{1/3} + \frac{1}{3}x^2y^{-1/3} = \frac{4}{3} \\
&y_3 = xy^{-1/3} - \frac{1}{9}x^3y^{-1} = \frac{8}{9} \\
&y_4 = y^{-1/3} - \frac{2}{3}x^2y^{-1} + \frac{1}{9}x^4y^{-5/3} = \frac{4}{9}
\end{aligned}$$

With Equation 61 one can then write down the following:

$$y(1.1) = 1 + \frac{1}{1!}h + \frac{\frac{4}{3}}{2!}h^2 + \frac{\frac{8}{9}}{3!}h^3 + \frac{\frac{4}{9}}{4!} = \underline{\underline{1.1068166666667}}$$

Then one does the same at the new location on got from the previous result.

$$\begin{aligned}
&x = 1.1, && y = 1.1068166666667 \\
&y(1.2) = && y + \frac{x \cdot y^{1/3}}{1!}h + \frac{y^{1/3} + \frac{1}{3}x^2y^{-1/3}}{2!}h^2 + \frac{xy^{-1/3} - \frac{1}{9}x^3y^{-1}}{3!}h^3 + \\
& && \frac{y^{-1/3} - \frac{2}{3}x^2y^{-1} + \frac{1}{9}x^4y^{-5/3}}{4!} = \underline{\underline{1.227872941753}}
\end{aligned}$$

The global error is defined by Equation 64 and therefore has the following result:

$$\begin{aligned}
&\max_{0 \le i \le 2} \left| y_i - y(x_i) \right| = \max\left\{ \left| y_0 - y(x_0) \right|, \left| y_1 - y(x_1) \right|, \left| y_2 - y(x_2) \right| \right\} = \\
&\max\left\{ |1-1|, \left| y_1 - \left(\frac{1.1^2+2}{3} \right)^{3/2} \right|, \left| y_2 - \left(\frac{1.2^2+2}{3} \right)^{3/2} \right| \right\} = \underline{\underline{1.1473\underline{4} \times 10^{-7}}}
\end{aligned}$$

The local error is defined by Equation 66 and therefore has the following result:

$$
\begin{aligned}
&n = 0\\
&\tau_h(x_0) := \frac{y(x_0+h) - y(x_0)}{h} - \left(\frac{y'(x_0)}{1!} + \frac{y''(x_0)}{2!}h^1 + \dots + \frac{y^{(4)}(x_0)}{4!}h^{4-1}\right) =\\
&\frac{y(1.1)-y(1)}{h} - \left(\frac{y'(1)}{1!} + \frac{y''(1)}{2!}h^1 + \dots + \frac{y^{(4)}(1)}{4!}h^{4-1}\right) =\\
&\frac{\left(\frac{1.1^2+2}{3}\right)^{3/2} - \left(\frac{1^2+2}{3}\right)^{3/2}}{0.1} - \left(1 + \frac{2}{3}0.1 + \frac{4}{27}0.1^2 + \frac{1}{54}0.1^3\right) = \underline{\underline{-6.035828648 \times 10^{-7}}}
\end{aligned}
$$

and for n=1 one has the following result:

$$
\begin{aligned}
\tau_h(x_1) := \frac{y(x_1+h) - y(x_1)}{h} - \left(\frac{y'(x_1)}{1!} + \frac{y''(x_1)}{2!}h^1 + \dots + \frac{y^{(4)}(x_1)}{4!}h^{4-1}\right) =\\
\frac{y(1.2)-y(1.1)}{h} - \left(\frac{y'(1.1)}{1!} + \frac{y''(1.1)}{2!}h^1 + \dots + \frac{y^{(4)}(1.1)}{4!}h^{4-1}\right) =\\
\frac{\left(\frac{1.2^2+2}{3}\right)^{3/2} - \left(\frac{1.1^2+2}{3}\right)^{3/2}}{0.1} - (\dots) = \underline{\underline{-5.2265445216 \times 10^{-7}}}
\end{aligned}
$$

8.3 Explicit Runge-Kutta Methods

Since the calculation of the tailor approximation series is quite tedious. One came up with another method, which is recursive and results in the same as the tailor series approximation (big advantage is that no computations of derivatives up to order p are needed). In Equation 68 one can find the Common Runge-Kutta method of order 4. As one can see there one has seven variables, when one now sets this equal with the tailor series approximation all those variables are defined in the end as it can be seen in Equation 69.

$$
\begin{aligned}
k_1 &= f(x,y)\\
k_2 &= f\left(x+mh, y+mhk_1\right)\\
k_3 &= f\left(x+nh, y+nhk_2\right)\\
k_4 &= f\left(x+ph, y+phk_3\right)\\
y(x+h) &\approx y(x) + ahk_1 + bhk_2 + chk_3 + dhk_4
\end{aligned}
\tag{68}
$$

Name der Lösungen	#Stages(s)	Lösungen
Heun	2	$a = b = \frac{1}{2}, m = 1 (n = p = c = d = 0)$
Explicit Midpoint	2	$a = 0, b = 1, m = \frac{1}{2} (n = p = c = d = 0)$
Classic Runge-Kutta	4	$m = n = \frac{1}{2}, p = 1, a = d = \frac{1}{6}, b = c = \frac{1}{3}$

(69)

For the classic runge-kutta method one can also write Equation 70 which results in Equation 71.

$$
\begin{aligned}
k_1 &= hf(x,y)\\
k_2 &= hf\left(x+mh, y+mk_1\right)\\
k_3 &= hf\left(x+nh, y+nk_2\right)\\
k_4 &= hf\left(x+ph, y+pk_3\right)\\
y(x+h) &\approx y(x) + ak_1 + bk_2 + ck_3 + dk_4
\end{aligned}
\tag{70}
$$

$$
\begin{aligned}
k_1 &= hf(x,y)\\
k_2 &= hf\left(x+\frac{1}{2}h, y+\frac{1}{2}k_1\right)\\
k_3 &= hf\left(x+\frac{1}{2}h, y+\frac{1}{2}k_2\right)\\
k_4 &= hf\left(x+h, y+k_3\right)\\
y(x+h) &\approx y(x)+\frac{1}{6}\left(k_1+2k_2+2k_3+k_4\right)
\end{aligned}
\tag{71}
$$

The errors are calculated nearly the same way as in the previous section. Equation 74.

- **Global Error:**

$$\max_{0\le i\le k}\left|y_i - y(x_i)\right| \tag{72}$$

- **Local Error:**

$$h\cdot\tau_h(x_n) \tag{73}$$

  When there is only one step calculated, the local error must be the same as the global error.

- **Local slope Error:**

$$\tau_h(x_n) := \frac{y(x_n+h)-y(x_n)}{h} - \left(\sum_{j=1}^{s} b_j k_j\right) \tag{74}$$

8.3.1 Example

Solve the initial value problem $y' = xy^{1/3} \quad y(1) = 1$ numerically by the classical Runge-Kutta method of order p = 4 and fixed step-size h = 0.1 for the x-values 1.1 und 1.2 (two steps). All final (!) results should be rounded to the 10th digit. The exact solution of the equation is $y = \left(\frac{x^2+2}{3}\right)^{3/2}$.
From Equation 71 one knows that:

$$
\begin{aligned}
k_1 &= hf(x,y) = hxy^{1/3} = 0.1\cdot 1\cdot 1^{1/3} = 0.1\\
k_2 &= hf\left(x+\frac{1}{2}h, y+\frac{1}{2}k_1\right) = 0.1\cdot(1+\frac{1}{2}\cdot 0.1)\cdot(1+\frac{1}{2}\cdot 0.1)^{1/3} = 0.10672161746556\\
k_3 &= hf\left(x+\frac{1}{2}h, y+\frac{1}{2}k_2\right) = 0.1\cdot(1+\frac{1}{2}\cdot 0.1)\cdot(1+\frac{1}{2}\cdot 0.10672161746556)^{1/3} = 0.10683535998891\\
k_4 &= hf\left(x+h, y+k_3\right) = 0.1\cdot(1+0.1)\cdot(1+0.10683535998891)^{1/3} = 0.11378552740653\\
y(x+h) &\approx y(x)+\frac{1}{6}\left(k_1+2k_2+2k_3+k_4\right) = \underline{\underline{1.1068165803859}}
\end{aligned}
$$

$$
\begin{aligned}
k_1 &= hf(x,y) = hxy^{1/3} = 0.1\cdot 1.1\cdot 1.1068165803859^{1/3} = 0.11378488387236\\
k_2 &= hf\left(x+\frac{1}{2}h, y+\frac{1}{2}k_1\right) = 0.1\cdot(1+\frac{1}{2}\cdot 0.1)\cdot(1.1+\frac{1}{2}\cdot 0.11378488387236)^{1/3} = 0.12096116868214\\
k_3 &= hf\left(x+\frac{1}{2}h, y+\frac{1}{2}k_2\right) = 0.1\cdot(1+\frac{1}{2}\cdot 0.1)\cdot(1.1+\frac{1}{2}\cdot 0.12096116868214)^{1/3} = 0.12108536369591\\
k_4 &= hf\left(x+h, y+k_3\right) = 0.1\cdot(1.1+0.1)\cdot(1.1068165803859+0.12108536369591)^{1/3} = 0.1284998068981\\
y(x+h) &\approx y(x)+\frac{1}{6}\left(k_1+2k_2+2k_3+k_4\right) = \underline{\underline{1.2278795396403}}
\end{aligned}
$$

As one can see, the result is nearly the same as in the previous Example. The **local slope error** can be calculated according to Equation 74.

$$\tau_h(x_0) := \frac{y(x_0+h)-y(x_0)}{h} - \left(\frac{\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4}{6h}\right) =$$

$$\frac{\left(\frac{1.1^2+2}{3}\right)^{3/2} - \left(\frac{1^2+2}{3}\right)^{3/2}}{0.1} - \left(\frac{\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4}{0.6}\right) = \underline{\underline{2.5922467994 \times 10^{-7}}}$$

$$\tau_h(x_1) := \frac{y(x_1+h)-y(x_1)}{h} - \left(\frac{\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4}{6h}\right) =$$

$$\frac{\left(\frac{1.2^2+2}{3}\right)^{3/2} - \left(\frac{1.1^2+2}{3}\right)^{3/2}}{0.1} - \left(\frac{\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4}{0.6}\right) = \underline{\underline{2.7090772847 \times 10^{-7}}}$$

The local error can be calculated according to Equation 73 and results in the following:

$$h \cdot \tau_h(x_0) = \underline{\underline{2.5922467994 \times 10^{-8}}}$$

$$h \cdot \tau_h(x_1) = \underline{\underline{2.7090772847 \times 10^{-8}}}$$

8.4 Butcher tableau

The Butcher tableau is mainly a mnemonic device (Gedächtnissstütze) to remember the coefficients. The tableau must fullfill the condition listed in Equation 75

$$\begin{aligned} & c_i = \sum_{j=1}^{s} a_{ij} = \sum_{j=1}^{i-1} a_{ij} (i = 2, \ldots, s) \\ & \sum_{j=1}^{s} b_j = 1 \\ & c_1 = 0 \\ & a_{1j} = 0 \quad 1 \le j \le s \\ & a_{ij} = 0 \quad j \ge i \end{aligned} \tag{75}$$

Where the variables mean the folwoing:

- s= number of stages
- c= 'inputs' Subdivides inputs
- b= weights
- a = subdiv 'y' 'output'

The general tableau can be seen in Equation 76.

$$\begin{array}{c|cccc} & k_1 & k_2 & \ldots & k_S \\ \hline c_1 & a_{11} & a_{12} & \ldots & a_{1s} \\ c_2 & a_{21} & a_{22} & \ldots & a_{2s} \\ \vdots & \vdots & \vdots & & \vdots \\ c_s & a_{S1} & a_{S2} & \ldots & a_{ss} \\ \hline & b_1 & b_2 & \ldots & b_S \end{array} \tag{76}$$

A step from x_n to $x_{n+1} = x_n + h_n (n = 0, 1, \ldots)$ in the general Runge-Kutta method is defined by Equation 77 where the values a, b, c can be read from the butcher tableau.

$$
\left.
\begin{aligned}
k_1 &= f\left(x_n + c_1 h_n, y_n + h_n \textstyle\sum_{j=1}^{s} a_{1,j} k_j\right) \\
k_2 &= f(x_n + c_2 h_n, \underbrace{y_n + h_n \sum_{j=1}^{s} a_{2,j} k_j}_{g_2}) \\
k_3 &= f(x_n + c_3 h_n, \underbrace{y_n + h_n \sum_{j=1}^{s} a_{3,j} k_j}_{g_3}) \\
&\vdots \\
k_s &= f(x_n + c_s h_n, \underbrace{y_n + h_n \sum_{j=1}^{s} a_{s,j} k_j}_{g_s})
\end{aligned}
\right\} \text{s stages}
\tag{77}
$$

$$y_{n+1} = y_n + h_n \textstyle\sum_{j=1}^{s} b_j k_j$$

Some examples for butcher tableaus can be found in Table 1

$\begin{array}{c\|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$	$\begin{array}{c\|c} 0 & \\ \hline & 1 \end{array}$ $\begin{array}{c\|cc} 0 & & \\ 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$	$\begin{array}{c\|cc} 0 & & \\ 2/3 & 2/3 & \\ \hline & 1/4 & 3/4 \end{array}$
Classical Runge-Kutta of order 4 (cf. 1.10c)	Explicit Euler method (cf. 1.3A') Explicit Midpoint method (1.9b)	Yet another method (cf. 1.9c)

Table 1: Butcher Tableaus

The simplest adaptive Runge–Kutta method involves combining Heun's method, which is order 2, with the Euler method, which is order 1 (also called Heun-Euler 2(1)). Its extended Butcher Tableau can be seen in Equation 78.

$$
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & 1/2 & 1/2 \\
 & 1 & 0
\end{array}
\tag{78}
$$

8.5 Step-size adaption

8.5.1 Idea

The Idea is to automatically adapt the step-size h. Due to that one needs a new way to define the approximation of the error, which can be done with an Accuracy Goal (ac), which defines how many decimal places (Nachkommastellen) are correct and a precision Goal (pg), which represents the significant digits of the result. The two parameters are considered in the tolerance parameter, ε which can be found in Equation 79.

$$\varepsilon = \varepsilon_a + |y|\varepsilon_r = 10^{-ag} + |y|10^{-pg} \geq |e| \tag{79}$$

Furthermore, one needs a second approximation for the error calculation, with the order $\hat{p}$. The first order approximation p is needed to calculate the step size. Mostly $\hat{p} = p - 1$. Due to that the butcher tableau is extended by a row (b values) as it can be seen in Table 2. The local error can be calculated according to

0	0	0	$\cdots$	0	0
c_2	$a_{2,1}$	0	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	0	$\vdots$
c_{s-1}	$a_{s-1,1}$	$a_{s-1,2}$	$\cdots$	0	0
c_s	$a_{s,1}$	$a_{s,2}$	$\cdots$	$a_{s,s-1}$	0
	b_1	b_2	$\cdots$	b_{s-1}	b_s
	$\hat{b}_1$	$\hat{b}_2$	$\cdots$	$\hat{b}_{s-1}$	$\hat{b}_s$

Table 2: Butcher Table

Equation 80.

$$e_n = y(x+h) - \hat{y}(x+h) = h_n \sum_{j=1}^{s} \left(b_j - \hat{b}_j\right) k_j \Rightarrow \|e_n\| = \left\| h_n \sum_{j=1}^{s} \left(b_j - \hat{b}_j\right) k_j \right\| \tag{80}$$

Below one can find again a short description of the variables:

- $\varepsilon_a = dy$: absolute error $\varepsilon_a = 10^{-ag}$
  Ex. $ag = 4 \Rightarrow 0.001289$
- $\varepsilon_r = \frac{dy}{y \neq 0}$: relative error $\varepsilon_r = 10^{-pg}$
  Ex. $pg = 4 \Rightarrow 0.00\underbrace{1289}_{\text{signific. digit}}$

To know if the step size is good or not one calculates $\frac{\|e_n\|}{\varepsilon}$. When the current step $\frac{\|e_n\|}{\varepsilon} > 1$, then the estimation of h_n was too optimistic and the step must be repeated with a smaller step size. One also says the current step is **Rejected**. Otherwise when $\frac{\|e_n\|}{\varepsilon} \leq 1$ the step size is ok and one can **proceed**. Updating the step size is done according to Equation 81.

$$h_{n+1} = h_n \left(\frac{\varepsilon}{\|e_n\|}\right)^{\frac{1}{\tilde{p}}} = h_n \left(\frac{\|e_n\|}{\varepsilon}\right)^{-\frac{1}{\tilde{p}}} \tag{81}$$

$\varepsilon = \varepsilon_a + \varepsilon_r \left|y_n\right|$ with $\tilde{p} = \min(p, \hat{p}) + 1$ (order of the primary method)

8.5.2 Stability of explicit methods

The global relative error must not diverge, which means it must be limited. Since it is difficult to make a statement about the analysed ODE one uses benchmark equations. One of the most commonly used ones is the Dahlquist model, which can be seen in Equation 82

$$y' = Ay \quad y(0) = 1 \quad \text{mit } A = \Re\{A\} + j\Im\{A\} \in \mathbb{C} \tag{82}$$

The solution of this equation is $y = e^{\Re\{A\}x}\left(\cos(\Im\{A\}x + j\sin(\Im\{A\}x)\right)$, which is an oscillation with a exponential amplitude $e^{\Re\{A\}x}$ and frequency $\Im\{A\}$. The

Example Euler

$$Y' = -\lambda Y; Y(0) = 1; x \geq 0; \lambda > 0$$

The exact solution is: $Y(x) = e^{-\lambda x}$ Consider Euler's method: Solation will go to zero iff

$$\begin{aligned} y_{n+1} &= y_n + hf\left(x_n, y_n\right) & &|1 - h\lambda| < 1 \Rightarrow \frac{2}{\lambda} > h > 0 \\ &= y_n - h\lambda y_n & &\text{Euler's method is stable for} \\ &= (1 - h\lambda)y_n & &\text{this ODE if} \quad 0 < h < \frac{2}{\lambda} \end{aligned}$$

Stability for heun-method (rk-2)
From Equation 77 one knows that $y_{k+1} = y_k + h \cdot (\frac{1}{2} \cdot k_1 + \frac{1}{2} \cdot k_2)$

$$\begin{aligned} k_1 &= Ay_k & y_0 &= y(x_0) = y(0) = 1 \\ k_2 &= A\left(y_k + 1hk_1\right) = A\left(y_k + Ahy_k\right) \quad \Longrightarrow & y_{k+1} &= y_k \underbrace{\left(1 + hA + (hA)^2\frac{1}{2}\right)}_{F(hA)=F(z), \quad z=hA\in\mathbb{C}} \end{aligned} \tag{83}$$

From that one can somehow derive three cases which are listed below

3 Cases

- $\Re\{A\} < 0$: The Amplitude of the ODE gets smaller exponentially. The stability conditions says that the approximated values $y_k(k = 0, 1, \ldots)$ must be exponentially damped too. Therefore and from $y_{k+1} \propto |F(z)|$ the following must hold $|F(z)| < 1$.
- $\Re\{A\} = 0$: The Amplitude of the ODE is constant. The stability conditions says that the approximated values $y_k(k = 0, 1, \ldots)$ are constant too. Therefore and from $y_{k+1} \propto |F(z)|$ the following must hold $|F(z)| = 1$.
- $\Re\{A\} > 0$: The Amplitude of the ODE gets larger exponentially. The stability conditions says that the approximated values $y_k(k = 0, 1, \ldots)$ must be exponentially growing too. Therefore and from $y_{k+1} \propto |F(z)|$ the following must hold $|F(z)| > 1$.

The stability condition for case one can know be calculated as the following ($\Re\{A\} < 0$) and $1 > |F(z)| = \left|1 + hA + \frac{1}{2}(hA)^2\right| \Rightarrow -2 < h\Re\{A\} < 0$. since $x_{1,2} = \frac{-b\pm\sqrt{(b^2-4\cdot a\cdot c)}}{2\cdot a} = \frac{-1\pm\sqrt{((1)^2-4\cdot 0\cdot\frac{1}{2})}}{2\cdot\frac{1}{2}} = -1 \pm 1$ and therefore the values must be between 0 and -2. The **stability polynomial** in this exercise was $F(z) = 1 + z + \frac{z^2}{2} \quad (z = hA)$.

Recursive Formulas
For the stability polynomials in the form: $F(z) = 1 + b_1k_1(z) + \ldots + b_sk_s(z)$ recursive equations exist as can be seen in Equation 84.

$$k_1(z) = z, \quad k_{j+1}(z) = z\left(1 + a_{j+1,1}k_1(z) + a_{j+1,2}k_2(z) + \ldots + a_{j+1,j}k_j(z)\right) \tag{84}$$

8.5.3 Exercise adaptive step size

Solve the initial value problem $\varphi' = c(1-\varepsilon\cos\varphi)^2 \quad \varphi(0)=0$ with $c=1$ and $\varepsilon = 0.25$ numerically by applying the Heun-Euler 2(1) embedded adaptive method with classical step-size control until 3 proceeding steps are executed. The initial step-size equals 0.001, the accuracy goal (ag) 4 and the precision goal is 4 , either. Create a table listing values for $\left(x, y, h, e_k, \left(\frac{\|e_k\|}{\varepsilon}\right)^{-\frac{1}{p}}, h_{\text{new}}\right.$, state) containing at least three preceding steps.

According to the exercise, we know the following:

1. The position at the beginning: $x=0,\ y=0$
2. The step size: $h = 0.001$
3. The local error can be calculated according to Equation 80. Which says that $e_k = h_n \sum_{j=1}^{s}\left(b_j - \hat{b}_j\right)k_j$. From Equation 78 one knows $b_1 = \frac{1}{2}$, $\hat{b}_1 = 1$, $b_2 = \frac{1}{2}$ and $\hat{b}_2 = 0$. Furthermore, one knows from Equation 77 and Equation 78 that $k_1 = f\left(x_n + \underbrace{0}_{c_1}\cdot h_n, y_n + h_n\cdot \underbrace{0}_{a_{1,j}}\right) = 1\cdot\left(1-\frac{1}{4}\cos 0\right)^2 = \frac{9}{16}$ and $k_2 = f\left(x_n + \underbrace{1}_{c_2}\cdot h_n, y_n + h_n \cdot \underbrace{1}_{a_{2,1}}\cdot k_1\right) = 1\cdot\left(1-\left(\frac{1}{4}\cdot\cos\left(0.001\cdot 1\cdot\frac{9}{16}\right)\right)\right)^2 = 0.5625$. Therefore $e_k = h_n\cdot(b_1-\hat{b}_1)\cdot k_1 + (b_2-\hat{b}_2)\cdot k_2 = \underbrace{0.001}_{h_n}\cdot\left(\underbrace{-\frac{1}{2}}_{b_1-\hat{b}_1}\cdot\frac{9}{16} + \underbrace{\frac{1}{2}}_{(b_2-\hat{b}_2)}\cdot 0.5625\right) = 2.966\cdot 10^{-11}$
4. Since $p = 2$ (order) and ε can be calculated according to Equation 79 which says: $\varepsilon = \varepsilon_a + |y|\varepsilon_r = 10^{-ag} + |y|10^{-pg} = 10^{-4} + 0\cdot 10^{-4} = 10^{-4}$. Therefore $\left(\frac{\|e_k\|}{\varepsilon}\right)^{-\frac{1}{p}} = \left(\frac{2.966\cdot 10^{-11}}{10^{-4}}\right)^{-\frac{1}{2}} = 1.836\cdot 10^3$
5. With Equation 81 one can the finally calculate the new step size which is: $h_{n+1} = h_n\left(\frac{\|e_n\|}{\varepsilon}\right)^{-\frac{1}{p}} = 0.001\cdot\left(\frac{2.966\cdot 10^{-11}}{10^{-4}}\right)^{-\frac{1}{2}} = 1.836$
6. the new y value can be calculated according to Equation 77 which means for the current scheme $0.001\cdot\left(\frac{1}{2}\cdot k_1 + \frac{1}{2}\cdot k_2\right) = 0.001\cdot\left(\frac{1}{2}\cdot\frac{9}{16} + \frac{1}{2}\cdot 0.5625\right) = 0.0005625$

x	y	h_n	e_k	$\left(\frac{\|e_k\|}{\varepsilon}\right)^{-\frac{1}{p}}$	h_{n+1}	state
0	0	0.001	$2.966\cdot 10^{-11}$	$1.836\cdot 10^3$	1.836	Proceed
0.001	0.0005625	1.836	0.18169	0.02347	0.043	Reject

Table 3: Exercise

8.5.4 Exercise Stability polynomial

Using Theorem 1.3 in the script (p. 29) gain the A-stability polynomial $F_1(z) = 1+z+\frac{z^2}{2}+\frac{z^3}{6}$ for the embedded adaptive method SS3(2) with Butcher tableau.

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ 1 & -1 & 2 & & \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ & \frac{1}{72}(\sqrt{82}-10) & \frac{1}{36}(10-\sqrt{82}) & \frac{1}{144}(28-\sqrt{82}) & \frac{1}{48}(\sqrt{82}-16) \end{array}$$

From Equation 84 one knows that a stability polynomial is of the form $F(z) = 1+b_1k_1(z)+b_2k_2(z)+b_3k_3(z)+0k_4(z)$, whereas:

$$k_1(z) = z$$

$$k_2(z) = z\left(1+\frac{1}{2}k_1(z)\right) = z\left(1+\frac{1}{2}z\right)$$

$$k_3(z) = z(1+-1k_1(z)+2k_2(z)) = z\left(1-z+2z\left(1+\frac{1}{2}z\right)\right) = z+z^2+z^3$$

$$\text{Therefore } F(z) = 1+\frac{1}{6}k_1(z)+\frac{2}{3}k_2(z)+\frac{1}{6}k_3(z) = 1+z+\frac{1}{2}z^2+\frac{1}{6}z^3$$

8.5.5 Stiffness

The stiffnes is dependent on:

- ODE
- step size h
- Method (for example RK-4)

Stiffness Detection

Heun euler does not work when we have stiffness situations. Condition: $c_{s-1} = c_s = 1$. Through testing of $\left|h\frac{\partial f(x,y)}{\partial y}\right|$ to the absolute borders of the stability region the stiffness can be detected. Stiffness is present when $|h\tilde{\lambda}|$ is outside or at the border of the stability condition.

Example for explicit runge-kutta

$$k_{s-1} = f(x+c_{s-1}h, \underbrace{y+h\sum_{j=1}^{s} a_{s-1,j}k_j}_{g_{s-1}}) \quad k_s = f(x+c_sh, \underbrace{y+h\sum_{j=1}^{s} a_{s,j}k_j}_{g_s}) \quad \Rightarrow \quad \tilde{\lambda} = \frac{\|k_s-k_{s-1}\|}{\|g_s-g_{s-1}\|}$$

$\tilde{\lambda}$ is an estimation for $f_y = \frac{\partial}{\partial y}f(x,y)$ for example $y' = x^4-25y^4 = f(x,y)$ and takes the role of $\Re\{A\}$ for the stability analysis.

8.5.6 Exercise stiffness detection test

Given the differential equation $y' = \underbrace{-\sqrt{x^2+y^2}}_{f(x,y)} \quad y(0)=4$ carry out a stiffness detection test using the A-stability region and the partial derivative f_y at the initial values and step size $h=1$. The method is defined by the Butcher tableau below.

$$\begin{array}{c|cc} 0 & & \\ 1/2 & 1/2 & \\ b & 0 & 1 \\ \hat{b} & 1 & 0 \\ \hline \end{array}$$

The exercise can be solved in two steps:

1. Get stability region (Equation 84):
   From Equation 84 one knows that a stability polynomial is of the form $\mathrm{F}(z) = 1 + b_1 k_1(z) + b_2 k_2(z) + b_3 k_3(z) + 0k_4(z)$, whereas:
   $$k_1(z) = z$$
   $$k_2(z) = z\left(1 + \frac{1}{2}k_1(z)\right) = z\left(1 + \frac{1}{2}z\right)$$
   $$\text{Therefore } \mathrm{F}(z) = 1 + 0k_1(z) + 1k_2(z) = 1 + z + \frac{1}{2}z^2 \text{ where } (z = h\cdot\mathrm{A})$$
   To calculate the stability region one hast to set the stability polynomial to zero and therefore one gets: $x_{1,2} = \frac{-b\pm\sqrt{(b^2-4\cdot a\cdot c)}}{2\cdot a} = \frac{-1\pm\sqrt{((1)^2-4\cdot 0\cdot\frac{1}{2})}}{2\cdot\frac{1}{2}} = -1\pm 1$ and therefore the boundary is -2 and 0.

2. Check stiffness (see subsubsection 8.5.5)
   By calculating the partial derivative of $-\sqrt{x^2+y^2}$ after y one obtains the following $-1\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x^2+y^2}}\cdot 2\cdot y$ inserting $x=0$ and $y=4$ results in $-1 \Rightarrow \mathrm{A}=-1$ and $h\cdot\mathrm{A}=-1$ which is inside the A-stability constraint $-2<\mathrm{A}h<0$ and therefore no stiffness is detected.

8.5.7 Van der Pol second-order differential equation

Solve the van-der-Pol ODE-system $\begin{pmatrix} z' = v \\ v' = \mu\left(1-z^2\right)v - z \end{pmatrix}$ $z(0)=1, v(0)=-1, \quad \mu=0.2$ numerically by applying the Heun-Euler 2(1) embedded adaptive method with classical step-size control until 3 proceeding steps are executed. The initial step-size equals 0.001, the accuracy goal (ag) 1 and the precision goal is 2. Create a table listing values for $\left(t, \{z,v\}, h, e_k, \left\|\frac{\vec{e}_n}{\varepsilon_a+\varepsilon_r\vec{y}_n}\right\|^{-1/\tilde{p}}, h_{\text{new}}, \text{ state}\right)$ containing at least three proceeding steps.

According to the exercise, we know the following:

1. The position at the beginning: $x=0$, $z=0$, $v=-1$
2. The step size: $h=0.001$
3. The local error can be calculated according to Equation 80. Which says that $e_k = h_n\sum_{j=1}^{s}\left(b_j-\hat{b}_j\right)k_j$. From Equation 78 one knows $b_1=\frac{1}{2}$, $\hat{b}_1=1$, $b_2=\frac{1}{2}$ and $\hat{b}_2=0$. Furthermore, one knows from Equation 77 and Equation 78 that $k_1 = f\left(x_n + \underbrace{0}_{c_1}\cdot h_n, y_n + h_n\cdot\underbrace{0}_{a_{1,j}}\right) = \begin{bmatrix} v \\ v' = \mu\left(1-z^2\right)v - z\end{bmatrix} = \begin{bmatrix} -1 \\ 0.2(1-1^2)\cdot(-1)\cdot \end{bmatrix}$

$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and

$$k_2 = f\left(x_n + \underbrace{0}_{c_2} \cdot h_n, y_n + h_n \cdot \underbrace{1}_{a_{2,1}} \cdot k_1\right)$$
$$= \begin{bmatrix} -1 + 0.001 \cdot 1 \cdot (-1) \\ 0.2(1-(1+0.001 \cdot 1 \cdot (-1))^2) \cdot (-1 + 0.001 \cdot 1 \cdot (-1)) - (1 + 0.001 \cdot 1 \cdot (-1)) \end{bmatrix}$$
$$= \begin{bmatrix} -1.001 \\ -0.9994001998 \end{bmatrix}$$

Therefore $e_k = 0.001 \cdot (-\frac{1}{2} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} -1.001 \\ -0.9994001998 \end{bmatrix}) = \begin{bmatrix} -5 \cdot 10^{-1} \\ 2.999001 \cdot 10^{-1} \end{bmatrix}$

4. Sine $p = 2$ (order) and ε can be calculated according to Equation 79 which says: $\varepsilon = \varepsilon_a + |y|\varepsilon_r = 10^{-ag} + |y|10^{-pg} = 10^{-1} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 10^{-2} = \begin{bmatrix} 0.11 \\ 0.09 \end{bmatrix}$. Therefore $\left(\frac{\|e_k\|}{\varepsilon}\right)^{-\frac{1}{p}} = \left(\max\left(\text{abs}\left(\begin{bmatrix} -5 \cdot 10^{-1} \\ 2.999001 \cdot 10^{-1} \end{bmatrix} ./ \left(\left(\begin{bmatrix} 0.11 \\ 0.09 \end{bmatrix}\right)\right)\right)\right)\right)$ 469.04157598235

5. With Equation 81 one can the finally calculate the new step size which is: $h_{n+1} = h_n \left(\frac{\|e_n\|}{\varepsilon}\right)^{-\frac{1}{p}} = 0.001 \cdot 469.04157598235 = 0.46904157598235$

6. the new y value can be calculated according to Equation 77 which means for the current scheme $0.001 \cdot (\frac{1}{2} \cdot k_1 + \frac{1}{2} \cdot k_2) = 0.001 \cdot (\frac{1}{2} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} -1.001 \\ 00.9994001998 \end{bmatrix}) = \begin{bmatrix} 0.9989995 \\ -1.0009997000999 \end{bmatrix}$

x	y	h_n	e_k	$\left(\frac{\|e_k\|}{\varepsilon}\right)^{-\frac{1}{p}}$	h_{n+1}	state
0	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	0.001	$\begin{bmatrix} -5 \cdot 10^{-1} \\ 2.999001 \cdot 10^{-1} \end{bmatrix}$	469.04157598235	0.46904	Proceed
0.001	$\begin{bmatrix} 0.9989995 \\ -1.0009997000999 \end{bmatrix}$	0.46904157598235				Reject

Table 4: Exercise

$$z''(t) = \underbrace{\mu\left(1 - z^2\right)}_{\text{non-linear damping}} z' - z \tag{85}$$

$$\begin{aligned} z' &= z_1 \\ z_1' &= z_2 \\ z_2{}' &= z_3 \\ &\vdots \\ z_{n-1}{}' &= f(x, z, z_1, z_2, \cdots, z_{n-1}) \end{aligned} \tag{86}$$

$$\begin{aligned} z' &= v \\ z_1' &= \mu\left(1 - z^2\right) z_1 - z \end{aligned} \tag{87}$$

$$\vec{y}''(x) = \frac{\overrightarrow{df}}{dx} = \mathrm{J}_{\vec{f}}(x,\vec{y}) \cdot \left(1, \vec{y}'(x)\right)^{tr} = (\frac{\partial \vec{f}}{\partial x}, \underbrace{\frac{\partial \vec{f}}{\partial y_1}, \frac{\partial \vec{f}}{\partial y_2}, \cdots, \frac{\partial \vec{f}}{\partial y_m}}_{=:\frac{\partial \vec{f}}{\partial \vec{y}} = \mathrm{J}_{\vec{f}}(\vec{y})}) \cdot (1, \underbrace{y_1'(x), y_2'(x), \cdots, y_m'(x))}_{=:y'(x)}^{tr} \tag{88}$$

$$= \frac{\partial \vec{f}}{\partial x} + \frac{\partial \vec{f}}{\partial \vec{y}} \cdot \vec{y}'(x) := \vec{f}_x + \vec{f}_{\dot{y}} \cdot \vec{f} := \vec{\mathrm{F}}_1$$

$$\vec{f}(t,z,v) = \begin{pmatrix} f_1(t,z,v) = & v \\ f_2(t,z,v) = & \mu\left(1-z^2\right)v - z \end{pmatrix} \tag{89}$$

	Zeitbereich	Frequenzbereich	
Linearity	$c_1 x_1(t) + c_2 x_2(t)$	$c_1 X_1(f) + c_2 X_2(f)$	$c_1 X_1(j\omega) + c_2 X_2(j\omega)$
Faltung	$x(t) * y(t)$	$X(f) \cdot Y(f)$	$X(j\omega) \cdot Y(j\omega)$
Multiplikation	$x(t) \cdot y(t)$	$X(f) * Y(f)$	$\frac{1}{2\pi} X(j\omega) * Y(j\omega)$
Verschiebung	$x(t - t_0)$	$X(f) \cdot e^{-j2\pi f t_0}$	$X(j\omega) \cdot e^{-j\omega t_0}$
Modulation	$e^{j2\pi f_0 t} \cdot x(t)$	$X\left(f - f_0\right)$	$X\left(j\left[\omega - \omega_0\right]\right)$
lineare Gewichtung	$t \cdot x(t)$	$-\frac{1}{j2\pi} \frac{d}{df} X(f)$	$-\frac{d}{d(j\omega)} X(j\omega)$
Differentiation	$\frac{d}{dt} x(t)$	$j2\pi f \cdot X(f)$	$j\omega \cdot X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\delta(f)$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(f)$
Skalierung	$x(at)$	$\frac{1}{\lvert a \rvert} \cdot X\left(\frac{f}{a}\right)$	$\frac{1}{\lvert a \rvert} \cdot X\left(\frac{j\omega}{a}\right)\delta(\omega)$
Zeitinversion	$x(-t)$	$X(-f)$	$X(-j\omega)$
konj. komplex	$x^*(t)$	$X^*(-f)$	$X^*(-j\omega)$
Real part	$x_R(t)$	$X_{g^*}(f)$	$X_{g^*}(j\omega)$
Imaginary part	$j x_I(t)$	$X_{u^*}(f)$	$X_{u^*}(j\omega)$
duality	$X(t)[X(jt)]$	$x(-f)$	$2\pi x(-\omega)$
Parsevalsches Theorem	$\int_{-\infty}^{\infty} x(t) \cdot y^*(t) dt = \int_{-\infty}^{\infty} X(f) \cdot Y^*(f) df = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot Y^*(j\omega) d\omega$		

Nr.	$x(t)$	$X(f)$	$X(j\omega)$
1	$\delta(t)$	1	1
2	1	$\delta(f)$	$2\pi\delta(\omega)$
3	$U_T(t)$	$\frac{1}{\lvert T \rvert} IH_{\frac{1}{T}}(f)$	$\frac{2\pi}{\lvert T \rvert} U_{\frac{2\pi}{T}}(\omega)$
4	$\varepsilon(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$	$\pi\delta(\omega) + \frac{1}{j\omega}$
5	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$	$\frac{2}{j\omega}$
6	$\frac{1}{\pi t}$	$-j\,\mathrm{sgn}(f)$	$-j\,\mathrm{sgn}(\omega)$
7	$\mathrm{rect}\left(\frac{t}{T}\right) \quad (T = width)$	$\lvert T \rvert \cdot \mathrm{si}(\pi T f)$	$\lvert T \rvert \cdot \mathrm{si}\left(\frac{T}{2}\omega\right)$
8	$\mathrm{si}\left(\pi \frac{t}{T}\right)$	$\lvert T \rvert \cdot \mathrm{rect}(T f)$	$\lvert T \rvert \cdot \mathrm{rect}\left(\frac{T}{2\pi}\omega\right)$
9	$\Lambda\left(\frac{t}{T}\right)$	$\lvert T \rvert \cdot \mathrm{si}^2(\pi T f)$	$\lvert T \rvert \cdot \mathrm{si}^2\left(\frac{T}{2}\omega\right)$
10	$\mathrm{si}^2\left(\pi \frac{t}{T}\right)$	$\lvert T \rvert \cdot \Lambda(T f)$	$\lvert T \rvert \cdot \Lambda\left(\frac{T}{2\pi}\omega\right)$
11	$e^{j2\pi f_0 t}$	$\delta\left(f - f_0\right)$	$2\pi\delta\left(\omega - \omega_0\right)$
12	$\cos\left(2\pi f_0 t\right)$	$\frac{1}{2}\left[\delta\left(f + f_0\right) + \delta\left(f - f_0\right)\right]$	$\pi\left[\delta\left(\omega + \omega_0\right) + \delta\left(\omega - \omega_0\right)\right]$
13	$\sin\left(2\pi f_0 t\right)$	$\frac{1}{2} j \left[\delta\left(f + f_0\right) - \delta\left(f - f_0\right)\right]$	$\pi j \left[\delta\left(\omega + \omega_0\right) - \delta\left(\omega - \omega_0\right)\right]$
14	$e^{-a^2 t^2}$	$\frac{\sqrt{\pi}}{a} e^{-\frac{\pi^2 f^2}{a^2}}$	$\frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}}$
15	$e^{-\frac{\lvert t \rvert}{T}}$	$\frac{2T}{1+(2\pi T f)^2}$	$\frac{2T}{1+(T\omega)^2}$

Where

$$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

$$\mathrm{si}(x) = \frac{\sin x}{x}$$

Common angles

Degrees	0°	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	

Reciprocal functions

$$\cot x = \frac{1}{\tan x}$$
$$\csc x = \frac{1}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$

Even/odd

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Pythagorean identities

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Cofunction identities

$$\sin(\frac{\pi}{2} - x) = \cos x$$
$$\cos(\frac{\pi}{2} - x) = \sin x$$
$$\tan(\frac{\pi}{2} - x) = \cot x$$
$$\cot(\frac{\pi}{2} - x) = \tan x$$
$$\sec(\frac{\pi}{2} - x) = \csc x$$
$$\csc(\frac{\pi}{2} - x) = \sec x$$

Sum and difference of angles

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double angles

$$\sin(2x) = 2 \sin x \cos x$$
$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$
$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

Half angles

$$\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$$
$$\cos \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}$$
$$\begin{aligned} \tan \frac{x}{2} &= \frac{1 - \cos x}{\sin x} \\ &= \frac{\sin x}{1 + \cos x} \end{aligned}$$

Power reducing formulas

$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$
$$\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

Product to sum

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$
$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$
$$\tan x \tan y = \frac{\tan x + \tan y}{\cot x + \cot y}$$
$$\tan x \cot y = \frac{\tan x + \cot y}{\cot x + \tan y}$$

Sum to product

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\tan x + \tan y = \frac{\sin(x+y)}{\cos x \cos y}$$
$$\tan x - \tan y = \frac{\sin(x-y)}{\cos x \cos y}$$

9 Fromulas

9.1 Differentation Formulas

1. $\frac{\mathrm{d}(u \pm v)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \pm \frac{\mathrm{d}v}{\mathrm{d}x}$
2. $\frac{\mathrm{d}(k \cdot u)}{\mathrm{d}x} = k \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$ (k konstant)
3. $\frac{\mathrm{d}(u \cdot v)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \cdot v + u \cdot \frac{\mathrm{d}v}{\mathrm{d}x}$
4. $\frac{\mathrm{d}(u/v)}{\mathrm{d}x} = \frac{\frac{\mathrm{d}u}{\mathrm{d}x} v - u\,\mathrm{d}v}{v^2}$
5. $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$ falls $z = f(y)$ und $y = g(x)$
6. $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = n x^{n-1}$
7. $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$
8. $\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a \quad (a > 0)$

9. $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$

10. $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$

11. $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$

12. $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \frac{1}{\cos^2 x}$

13. $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

14. $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}$

9.2 Integration Formulas

1. $\int_a^b (u \pm v)\mathrm{d}x = \int_a^b u\,\mathrm{d}x \pm \int_a^b v\,\mathrm{d}x$

2. $\int_a^b k \cdot u\,\mathrm{d}x = k\int_a^b u\,\mathrm{d}x \quad (k \text{ konstant })$

3. $\int_{x=a}^b f(g(x))g'(x)\mathrm{d}x = \int_{w=g(a)}^{g(b)} f(w)\mathrm{d}w$

4. $\int_a^b u \cdot \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = (u \cdot v)|_a^b - \int_a^b \frac{\mathrm{d}u}{\mathrm{d}x} \cdot v\,\mathrm{d}x$

9.3 Table of Indefinite Integrals

9.3.1 Basic Functions

1. $\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + \mathrm{C}, n \neq -1$

2. $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + \mathrm{C}$

3. $\int a^x\,\mathrm{d}x = \frac{1}{\ln a}a^x + \mathrm{C}, a > 0$

4. $\int \ln x\,\mathrm{d}x = x\ln x - x + \mathrm{C}$

5. $\int \sin x\,\mathrm{d}x = -\cos x + \mathrm{C}$

6. $\int \cos x\,\mathrm{d}x = \sin x + \mathrm{C}$

7. $\int \tan x\,\mathrm{d}x = -\ln|\cos x| + \mathrm{C}$

9.3.2 Products of e^x and cox x and sin x

8. $$\int \mathrm{e}^{ax}\sin(bx)\mathrm{d}x = \frac{1}{a^2+b^2}\mathrm{e}^{ax}(a\sin(bx) - b\cos(bx)) + \mathrm{C}$$

9. $$\int \mathrm{e}^{ax}\cos(bx)\mathrm{d}x = \frac{1}{a^2+b^2}\mathrm{e}^{ax}(a\cos(bx) + b\sin(bx)) + \mathrm{C}$$

10. $$\int \sin(ax)\sin(bx)\mathrm{d}x = \frac{1}{b^2-a^2}(a\cos(ax)\sin(bx) - b\sin(ax)\cos(bx)) + \mathrm{C}, a \neq b$$

11. $$\int \cos(ax)\cos(bx)\mathrm{d}x = \frac{1}{b^2-a^2}(b\cos(ax)\sin(bx) - a\sin(ax)\cos(bx)) + \mathrm{C}, a \neq b$$

12. $$\int \sin(ax)\cos(bx)\mathrm{d}x = \frac{1}{b^2-a^2}(b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)) + \mathrm{C}, a \neq b$$

9.3.3 Product of Polynomial $p(x)$ with $\ln x, \mathrm{e}^x, \cos x, \sin x$

13. $$\int x^n \ln x\,\mathrm{d}x = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + \mathrm{C}, n \neq -1$$

14. $$\int p(x)\mathrm{e}^{ax}\,\mathrm{d}x = \frac{1}{a}p(x)\mathrm{e}^{ax} - \frac{1}{a}\int p'(x)\mathrm{e}^{ax}\,\mathrm{d}x = \frac{1}{a}p(x)\mathrm{e}^{ax} - \frac{1}{a^2}p'(x)\mathrm{e}^{ax} + \frac{1}{a^3}p''(x)\mathrm{e}^{ax} - \ldots$$
(Sign alternate: $+-+-+-\ldots$)

15. $$\begin{aligned}\int p(x)\sin(ax)\mathrm{d}x &= -\frac{1}{a}p(x)\cos(ax) + \frac{1}{a}\int p'(x)\cos(ax)\mathrm{d}x \\ &= -\frac{1}{a}p(x)\cos(ax) + \frac{1}{a^2}p'(x)\sin(ax) + \frac{1}{a^3}p''(x)\cos(ax) - \ldots\end{aligned}$$
(Sign alternate in pairs after first term: $-++-++\ldots$)

16. $$\int p(x)\cos(ax)\mathrm{d}x = \frac{1}{a}p(x)\sin(ax) - \frac{1}{a}\int p'(x)\sin(ax)\mathrm{d}x$$
$$= \frac{1}{a}p(x)\sin(ax) + \frac{1}{a^2}p'(x)\cos(ax) - \frac{1}{a^3}p''(x)\sin(ax) - \ldots$$
(Signs alternate in pairs: $++--++-\ldots$)

9.4 Taylor Polynomial/Series

Development of f around a

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$
$$f(a+h) \approx f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \ldots$$

in which $k! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot k$.

9.4.1 Important Taylor Series

$$\mathrm{e}^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$$
$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$$
$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots \text{ (Geometric Series)}$$
$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

The last three only converge for $|x| < 1$.

9.5 Determinant

9.5.1 Sarrus

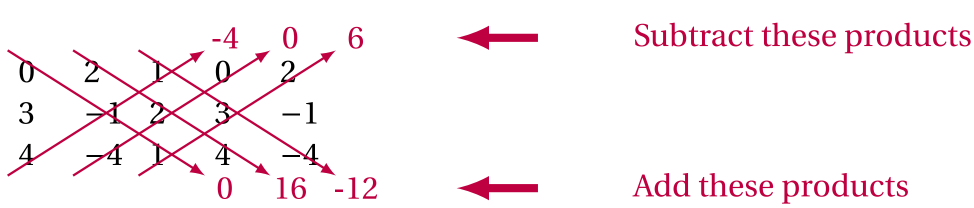

Figure 7: sarrus

9.6 Matrix

9.6.1 Transpose

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{3\times 2} \tag{90}$$

9.6.2 Multiplication

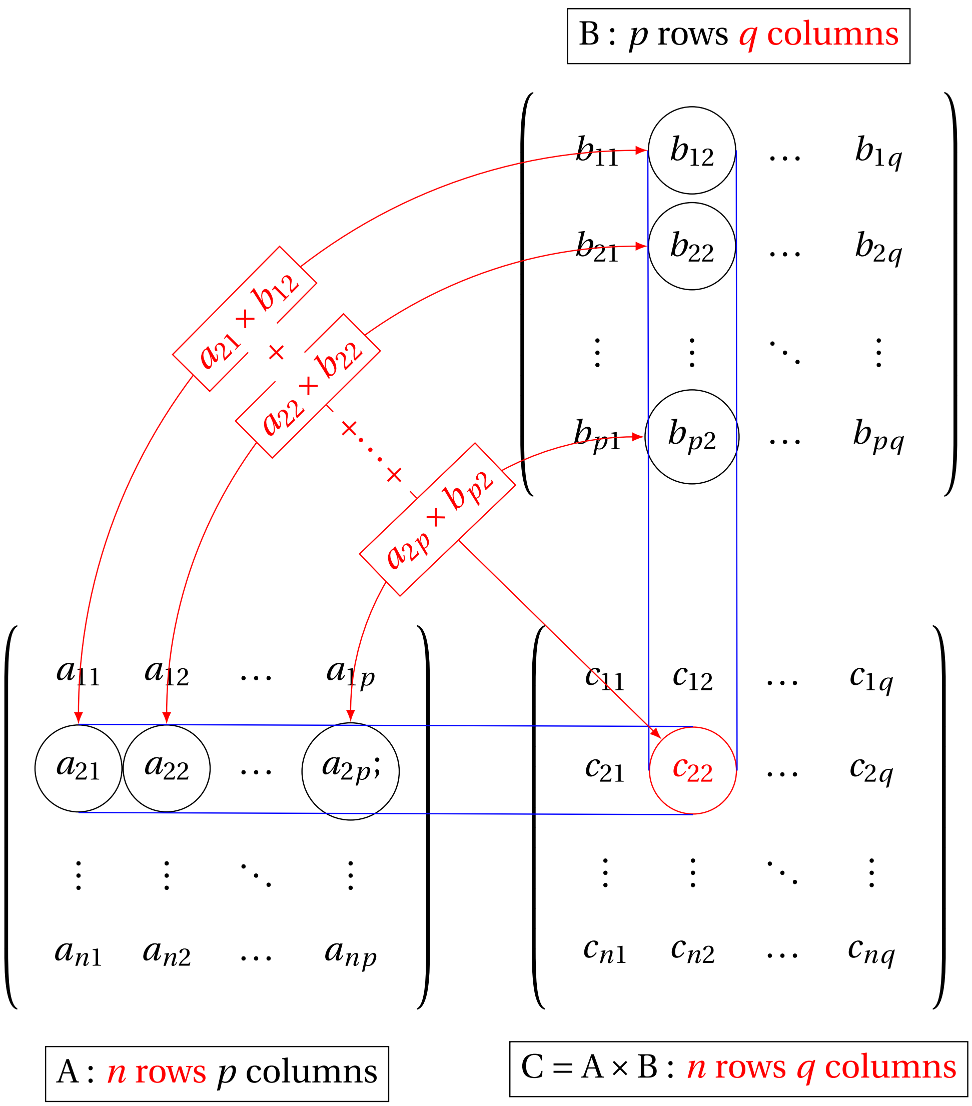

Figure 8: Matrix Multiplication